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GEOMETRY IN SPACE

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Clarendon Press Series

GEOMETRY IN SPACE

CONTAINING

PARTS OF EUCLID'S ELEVENTH AND TWELFTH BOOKS

AND SOME PROPERTIES OF POLYHEDRA AND SOLIDS OF REVOLUTION, WITH EXERCISES

EDITED BY

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PREFACE.

This little volume would have had its natural place as Part III of *Euclid Revised*, had not that Book extended to a quite sufficient size in its present form. But, mindful of the maxim—'Great book, great evil'—the Editor thought that it would be better to make a separate volume of those parts of Euclid's Eleventh and Twelfth Books, which are, by common consent, desirable for a student; with such analogous additional matter as it might seem advisable to treat from a purely geometrical standpoint. There is too the additional reason for taking this course, that the number of those who extend the study of Geometry from two to three dimensions is few compared with the number who content themselves with what can be done in one plane. No doubt the conception of solid figures, beyond the very simplest, is difficult; and the graphic exhibition of them is perhaps more difficult still. Yet there can be no doubt also that the study of Pure Geometry in Space is of immense value, whether considered as a mental discipline, or with regard to its utility for other purposes.

Respecting the range and amount of matter that should be included in a treatise designed to give an introduction to the purely geometrical treatment of figures in space, the Editor has had difficulty in coming to a decision; and he would put forward what is here given as, to a considerable extent, tentative. Thus, for example, it was originally intended that there should be a separate chapter on Surface Spherics; but, on reflection, it appeared dubious whether such a chapter would not be a 'game not worth the candle'—adding to the expense of the book, and not really wanted.

The consideration of these figures is therefore barely touched on, in a few Definitions and Exercises.

The utmost brevity of expression in the reasoning has been aimed at. Condensation, valuable everywhere, is peculiarly valuable here; for the subject has a natural tendency to prolixity; and it is easy to extend its proofs to an appalling length. Indeed it is likely that the extreme tedium, which a too strict adherence to the syllogistic form, in the exhibition of its reasonings, inevitably involves, has been a main cause of the neglect of Pure Geometry in Space.

Among treatises of a similar character the Editor's obligations are chiefly due to the *Géométrie dans l'Espace, par Rouché et De Comberousse;* and to the *Solid Geometry*, composed by the late Professor De Morgan, and published by the Society for the Diffusion of Useful Knowledge. With Mr. Wilson's *Solid Geometry* he has long been familiar, but he has not consciously made any use of that work; and where points of resemblance exist between it and his own, they are due (he believes) to some common source from which the two have drawn.

The Editor has to thank his Cambridge friends (*quondam* pupils), Mr. C. V. Coates, of Trinity College, and Mr. Alexander Larmor, of Clare College—the former for reading the proof sheets, and the latter for writing the Appendix on *The Geometrical Theory of Perspective in Space*, based on his own Paper in *The Quarterly Journal of Mathematics: No.* 84, 1886.

Opinions on the work, and especially on the advisability of expanding or contracting it, will be thankfully received.

ROYAL ACADEMICAL INSTITUTION, BELFAST.
January, 1888.

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ERRATUM.

In all cases, for frustrum *read* frustum

Nixon's Geometry in Space.

CHAPTER I.

PLANES AND SOLID ANGLES.

PRELIMINARY.

*Def*ʹ—The geometrical conception of position with the two extensions of length and breadth, but *not* the extension of thickness, is called **surface**; and when a surface is such that the join of *every* pair of points in it lies wholly on that surface, the surface is called a **plane**.

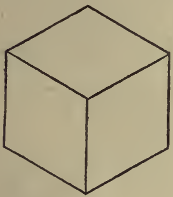
Note—If a given line and plane are sufficiently extended, then their relative positions will afford three alternatives: the line must—

1°, lie wholly in the plane; or
2°, meet it in one point only; or
3°, not meet it at all.

In other words, with respect to any plane, every line is in it, or cuts it, or is parallel to it.

In Plane Geometry the figures are all supposed to be in one plane: the consideration of figures *not* all in one plane gives rise to *Geometry in Space*. In the latter Science the figures may have the *three* extensions of length, breadth and thickness; and then they are said to be **solid**, or of **three dimensions**.

Note—On drawing the diagrams of solid figures.



If, in the graphic representation of a solid by a diagram, we make those lines which are equal in the solid also equal in its picture, the diagram is said to be *isometrical*. For example, the adjoining figure is the isometrical representation of a cube.

Though such a diagram may be useful (or even indispensable) in cases where it has to be used as a plan for giving proportional measurement, it does not give the idea of the object as it appears to the eye. For to the eye receding lines are, as it is called, *foreshortened*; and, when this foreshortened appearance is truly represented, the diagram is said to be drawn *in perspective*.

Without going into all the elaborate processes needed for the complete perspective drawing of any solid, there are a few simple rules, whose aid will be found very efficacious as a means of getting to a fair perspective approximation in such simple cases as will occur in the present work.

It will be necessary to premise certain technical terms. These will be better understood after some familiarity with the subject of the work has been attained.

Def—*The vertical line*, at any place, is the direction of a plumb-line at that place; and all planes through the vertical line are *vertical planes.*

Def—*A horizontal plane*, at any place, is any plane parallel to the surface of still water at that place; and all lines in a horizontal plane are *horizontal lines.*

Of course all horizontal planes, at any place, are perpendicular to the vertical line at that place.

If a solid, in any position, has lines drawn from all visible points on it to the eye, these lines are considered to be the *rays* by which it is seen. When the eye is fixed, all rays, simultaneously entering it, form a cone of rays, whose vertical angle is about 60°. Any plane across this cone of rays, and perpendicular to its axis, may be taken as the *picture-plane;* and the points in which the rays are cut by the picture-plane form the *image* of the solid on the picture-plane. The business of perspective drawing is to give a true representation of this image.

Def—The foot of the perpendicular from the eye on the picture-plane is called the *centre of vision;* and the line, in the picture-plane, drawn horizontally through the centre of vision is called *the horizontal line.*

The following are rules of perspective drawing—

Rule 1—All vertical lines in the object are vertical in its image.

Rule 2—All lines in the object, which are parallel to the picture-plane, retain their original relations and forms in its image: e.g. parallels, right angles, and equal lines, which in the object are parallel to the picture-plane, remain respectively parallels, right angles, and equal lines, in its image.

Rule 3—All lines in the object which are parallel to each other, but *not* parallel to the picture-plane, converge to points: these points are called *vanishing points.*

Rule 4—All lines in the object, which are perpendicular to the picture-plane, have, in its image, the centre of vision as vanishing point.

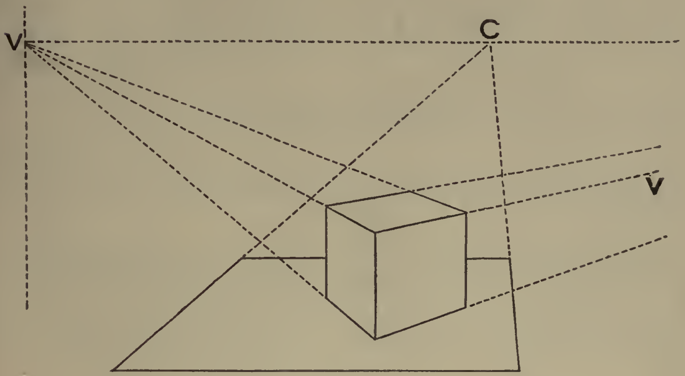
Rule 5—All parallel horizontal lines in the object, have, in its image, their vanishing points in the horizontal line.

Rule 6—All parallel horizontal planes in the object, converge, in its image, to the horizontal line.

Rule 7—All parallel vertical planes in the object, converge, in its image, to lines perpendicular to the horizontal line.

Rule 8—Circles in the object are, in its image, ellipses.

These rules are exemplified in the following diagram, representing a cube set sideways on a horizontal rectangular plane—the right and left hand edges of the plane being perpendicular to the picture-plane.



C is the centre of vision; **V** are the two vanishing points of the edges of the cube.

In looking at the diagram, hold the page vertically, and so that **C** is the foot of the perpendicular from the eye on the page.

In the subsequent diagrams, planes will sometimes be considered as transparent, and sometimes as opaque: in the latter case the occult lines will occasionally be indicated by dotted lines. The mode of representation chosen will be that which seems to render the diagram most perspicuous. It will be open to the reader to treat the occult lines otherwise, if he find his profit in so doing.

The Abbreviations used in this volume, additional to those already introduced in *Euclid Revised*, are—

1°, the symbol ||$^{\text{ped}}$, for parallelepiped; and

2°, some verbal contractions, whose meanings will be obvious from the context in which they occur: *e.g.* pl′, surf′, vol′, cyl′, pyr′, tet′, will stand respectively for plane, surface, volume, cylinder, pyramid, tetrahedron.

E·R· means *Euclid Revised.*

EUCLID—BOOK xi.

Proposition 1.

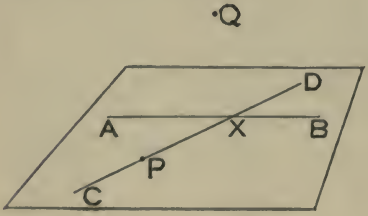
THEOREM—*If one part of a straight line is in a plane, all parts of it are in the same plane.*

As this property of a straight line is repeatedly assumed throughout Euclid's Plane Geometry, it seems absurd to give a formal proof of it here. In fact this so-called Theorem is involved in the idea of straightness, which is that—*If two straight lines coincide in any two points they coincide throughout their entire lengths.*

Ax'—If a plane, supposed to be indefinitely extended, is turned round any fixed straight line in itself, as a hinge, until it returns to its original position, it will, in its rotation, have passed through every point in space.

Proposition 2.

THEOREM—*Two intersecting straight lines are in one, and only one, plane.*



Let AB, CD be two st' lines cutting in X. Take any p't P in CD, and let any pl' thro' AB be turned round AB as a hinge until it passes thro' P.

Then ∵ **P** and **X** are in this pl′,

∴ the line **CXD** is in it.

Now when the pl′ contains **P**, let **Q** be any other p′t *not* in it. Then to make it contain **Q** it would have to be rotated again round **AB**.

But by this rotation it would cease to contain **P**.

∴ there is only one position of the pl′ in which it contains **AB** and **CD**:

i.e. **AB**, **CD** are in one, and only one pl′.

Note (1)—Since when the lines **AXB**, **CXD** are fixed the plane containing them is fixed, we may speak of a plane so determined as *the plane* **AXC**.

Note (2)—A plane is determined by—

1°, two intersecting lines; or
2°, a line, and a point not in it; or
3°, three points not collinear; or
4°, two parallel lines.

Proposition 3.

THEOREM—*If two planes cut one another their common section is a straight line.*

For let **P** be any p′t common to the pl′s;

and **Q** any other ,, ,,

Join **PQ**.

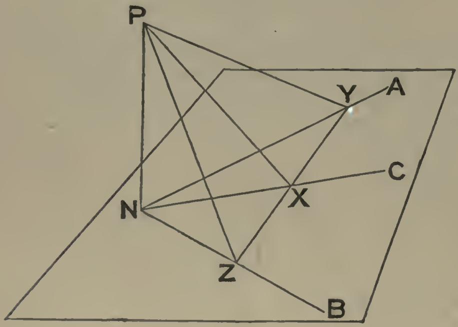
Then every p′t in **PQ** (and its production both ways) is wholly in one pl′, and wholly in the other:

i.e. **PQ** is the common section of the pl′s.

Note—Three planes will, in general, have one point in common: the exceptional case is when they intersect, two and two, in three parallel lines. Cf. Prop. 9.

Proposition 4.

THEOREM—*If a straight line is perpendicular to each of two intersecting lines, at their point of section, then it is also perpendicular to every line, which is in the plane determined by the intersecting lines, and which it meets.*



Let NA, NB be two intersecting st′ lines; PN ⊥ to each of them at N.

Draw any st′ line NC in the pl′ of $A\hat{N}B$.

In NC take any p′t X; and draw YXZ to meet NA, NB in Y, Z respectively, and so that XY = XZ.

Join PX, PY, PZ.

Then $2PX^2 + 2XY^2$

$= PY^2 + PZ^2$, ∵ PX is a median of $\triangle PYZ$,

$= 2PN^2 + NY^2 + NZ^2$, ∵ $\wedge^s$ PNY, PNZ are r′t,

$= 2PN^2 + 2NX^2 + 2XY^2$, ∵ NX is a median of $\triangle YNZ$,

∴ $PX^2 = PN^2 + NX^2$.

∴ $P\hat{N}X$ is right:

i.e. PX is ⊥ to any line (as NC) drawn from N in pl′ ANB.

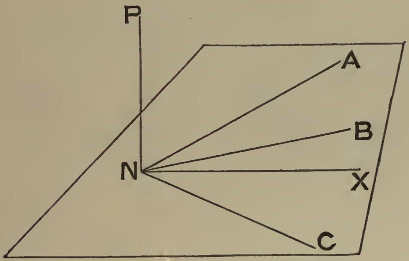
Def′—A straight line is said to be **perpendicular to a** plane, when it is perpendicular to every line which is in that plane, and which it meets; and the point of meeting is called the *foot* of the perpendicular.

Note—Hence Prop′ 4 may be worded thus—*If a line is perpendicular to each of two intersecting lines, at their point of section, it is perpendicular to their plane.*

Def′—Lines or points which are in the same plane are said to be **co-planar.**

Proposition 5.

THEOREM—*If a straight line is perpendicular to each of three concurrent lines, at their point of concurrence, then the three lines are co-planar.*



Let PN be ⊥ to each of the three concurrent lines NA, NB, NC, at N.

Let the pl′ PNB cut the pl′ ANC in NX.

Then PN, being ⊥ to NA, NC, is also ⊥ to NX.

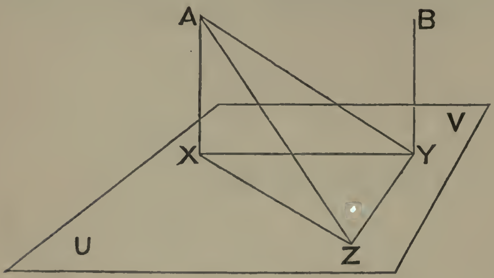
∴ $P\hat{N}X$ is r′t, and ∴ = $P\hat{N}B$.

But this is impossible unless NB, NX are coincident.

∴ NB is in the pl′ ANC.

Proposition 6.

THEOREM—*Straight lines which are perpendicular to the same plane are parallel.*



Let AX, BY be each ⊥ to the pl′ UV at X, Y respectively.

In the pl′ UV draw YZ ⊥ to YX, and equal to XA; and join XZ, AY, AZ.

Then in the $\triangle^s$ AXY, ZYX, since

$$\left.\begin{array}{r}\mathsf{AX} = \mathsf{ZY},\\ \mathsf{XY}\text{ is common,}\\ \text{and } \mathsf{A\hat{X}Y} = \mathsf{Z\hat{Y}X};\end{array}\right\}\text{ for each is r′t}$$

$$\therefore\ \mathsf{AY} = \mathsf{ZX}.$$

And since, in $\triangle^s$ AYZ, ZXA, we have also

AZ common, and YZ = XA;

$$\therefore\ \mathsf{A\hat{Y}Z} = \mathsf{Z\hat{X}A},$$

i.e. = a r′t ∧ ; ∵ AX is ⊥ to pl′ XYZ.

∴ the three lines BY, AY, XY, being each ⊥ to YZ, are co-planar.

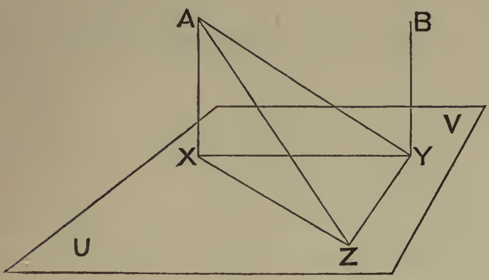
But AX is co-planar with AY, XY.

∴ AX, BY are co-planar;

and ∴, being ⊥ to XY, are ∥.

Proposition 8.

THEOREM—*If two straight lines are parallel, then if one of them is perpendicular to a plane, the other is perpendicular to the same plane.*



Let AX, BY be $\parallel^s$; and let the pl′ UV, to which BY is $\perp$ at Y, meet AX in X.

In the pl′ UV draw YZ $\perp$ to YX, and equal to XA; and join XZ, AY, AZ.

Then $B\hat{Y}X$ is r′t, $\because$ BY is $\perp$ to pl′ XYZ.

$\therefore$, as BY, AX are $\parallel^s$, $A\hat{X}Y$ is also r′t.

And $\because$ ZY is $\perp$ to YX, YB,

$\therefore$ ZY is $\perp$ to YA, which is in pl′ BYX, and meets it.

So that in $\triangle^s$ AXY, ZYX, we have

$$\left.\begin{aligned} AX &= ZY, \\ XY &\text{ common,} \\ \text{and } A\hat{X}Y &= Z\hat{Y}X; \end{aligned}\right\}$$

$$\therefore\ AY = ZX.$$

And since in $\triangle^s$ AXZ, ZYA, we have also

AZ common, and $YZ = XA$:

$$\therefore\ A\hat{X}Z = Z\hat{Y}A, \text{ a r't } \wedge:$$

i.e. AX is $\perp$ to XY and XZ;

and $\therefore$ is $\perp$ to the pl′ XYZ, to which BY is $\perp$.

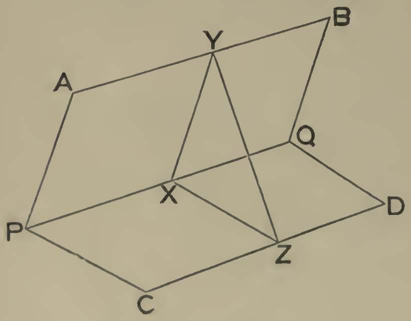
Proposition 7.

THEOREM—*If any two points are taken, one in each of two parallel straight lines, their join is co-planar with the parallels.*

This is perfectly obvious, and has been frequently assumed by Euclid.

Proposition 9.

THEOREM—*If two straight lines are each parallel to the same straight line, though the three are not co-planar, then the two are parallel to each other.*



Let AB, CD be st′ lines, each of which is ∥ to PQ — the three not being co-planar.

Take any p′t X in PQ; and draw XY, XZ, each ⊥ to PQ; so that XY is in pl′ of AB, PQ; and XZ in pl′ of CD, PQ—Y being on AB, and Z on CD. Join YZ.

Then PQ, being ⊥ to XY and XZ, is ⊥ to pl′ XYZ.

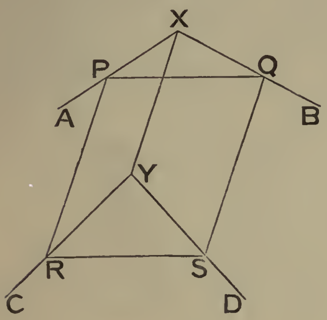
∴ AB, being ∥ to PQ, is also ⊥ to pl′ XYZ.

Sim′ly CD is ⊥ to pl′ XYZ.

∴ AB, CD, being ⊥ to the same pl′, are ∥.

Proposition 10.

THEOREM—*If a pair of concurrent lines are parallel to another pair of concurrent lines—the two pairs being in different planes—then the angle between the first pair is equal to the angle between the second pair; where both angles are taken of the same species.*



Let the lines XA, XB be ∥ to the lines YC, YD; but the pl's XAB, YCD be different.

Take any p't P in XA, and any p't Q in XB; and take R in YC, and S in YD, so that YR = XP, and YS = XQ.

Join PQ, RS.

Then ∵ PX is equal and ∥ to YR,
∴ PR „ „ XY.
Sim'ly QS „ „ XY.
∴ PR „ „ QS;
and ∴ also PQ „ „ RS.

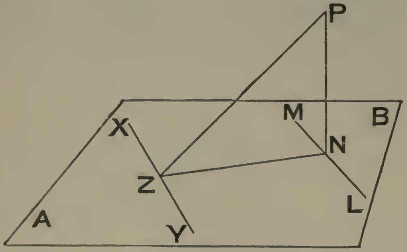
So that in $\triangle$s PXQ, RYS, we have

$$\left.\begin{array}{r} PX = RY, \\ XQ = YS, \\ \text{and}\quad PQ = RS; \end{array}\right\}$$

$$\therefore\quad P\hat{X}Q = R\hat{Y}S.$$

Proposition 11.

PROBLEM—*To draw a perpendicular to a given plane from a given point without it.*



Let **P** be a given p′t outside a given pl′ **AB**.

Take any line **XY** in **AB**; and draw **PZ** ⊥ to **XY**.

In the pl′ **AB** draw **ZN** ⊥ **XY**; and draw **PN** ⊥ to **ZN**.

If it happens that **PN** coincides with **PZ**, then it is the req′d ⊥. But if not—thro′ **N** draw **LNM** in the pl′ **AB**, and ‖ to **XY**.

Then since **XY** is ⊥ to **ZP** and to **ZN**,

∴ **XY** is ⊥ to the pl′ **PZN**.

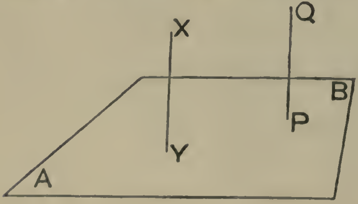
∴ **LM**, which is ‖ to **XY**, also is ⊥ to pl′ **PZN**.

∴ **PN** is ⊥ to both **NL** and **NZ**:

i.e. **PN** has been drawn ⊥ to pl′ **AB**.

Proposition 12.

PROBLEM—*To draw a line at right angles to a given plane from a given point in the plane.*



Let **P** be a given p′t in a given pl′ **AB**.

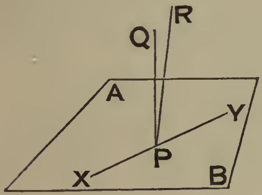
Take any p′t **X** outside the pl′; and drop **XY** ⊥ to the pl′.

Draw **PQ** ‖ to **XY**.

Then since of the $\|^s$ PQ, XY,
one XY is $\perp$ to pl′ AB,
$\therefore$ the other PQ is also $\perp$ to pl′ AB:
i.e. PQ has been drawn $\perp$ to pl′ AB.

Proposition 13.

THEOREM—*Through one point there can be drawn only one perpendicular to a given plane, whether the point is in, or outside, the plane.*



Let AB be a given pl′.
1°, take P any p′t *in* it.

Assume that two $\perp^s$ PQ, PR, can be drawn, at p′t P, to pl′ AB.

Let pl′ cont′g PQ, PR cut pl′ AB in XPY.

Then PQ, PR are each $\perp$ to XPY:

i.e. $Q\hat{P}Y = R\hat{P}Y$, a part of itself.

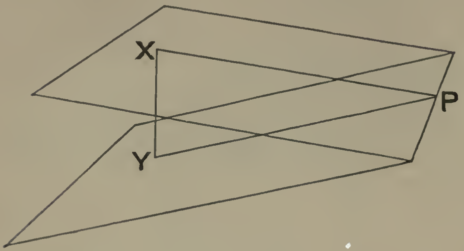
$\therefore$ the assumption that there can be two $\perp^s$ leads to an absurdity; and $\therefore$ is not true.

Nor, 2°, can two $\perp^s$ be drawn from a p′t *without* the pl′; for any two $\perp^s$ to a pl′ are $\|$, and $\therefore$ cannot have a common p′t.

Def′—Planes which cannot by production be made to meet are called **parallel**.

Proposition 14.

THEOREM—*Planes to which the same straight line is perpendicular are parallel.*



Let the st′ line **XY** be ⊥ to two pl′s at the p′ts **X**, **Y**: then if the pl′s could be produced to meet their common section would be a st′ line; and then, joining any p′t **P**, in this line of section, to **X** and **Y**, we should get a $\triangle$ having $P\hat{X}Y$ and $P\hat{Y}X$ each r′t.

But this is impossible.

∴ the pl′s cannot be produced to meet;

i.e. they are ∥.

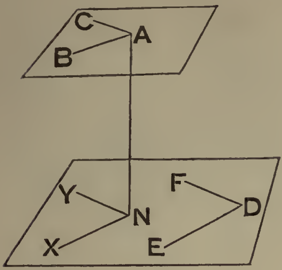
Proposition 15.

THEOREM—*If a pair of concurrent lines are parallel to another pair of concurrent lines—the pairs being in different planes—then the plane of the one pair is parallel to the plane of the other pair.*

Let the pair of concurrent lines **AB**, **AC** be respectively ∥ to the pair **DE**, **DF**.

Draw **AN** ⊥ to the pl′ **EDF**; and draw **NX**, **NY** respectively ∥ to **DE**, **DF**.

Then **AB** is ∥ to **NX**, ∵ each of them is ∥ to **DE**.



∴ $B\hat{A}N + X\hat{N}A$ = two r't $\wedge^{s}$.

But $X\hat{N}A$ is a r't ∧,

∴ $B\hat{A}N$ „

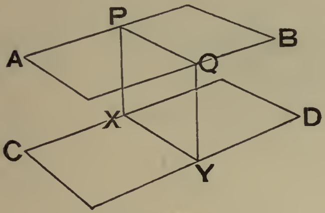
Sim'ly. $C\hat{A}N$ „

∴ NA is ⊥ to the pl' BAC.

∴ the pl's, being each ⊥ to NA, are ∥.

Proposition 16.

THEOREM—*If parallel planes are cut by a plane their sections with it are parallel.*

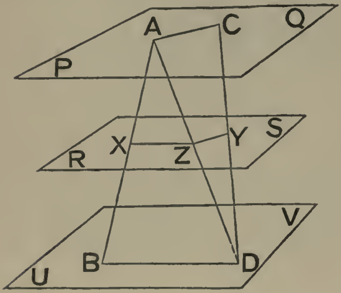


Let the ∥ pl's AB, CD be cut by any pl'—their respective lines of section being PQ, XY.

Then, if PQ, XY could be produced to meet—in Z say—
Z would be in each of the pl's AB, CD :
i.e. the pl's would meet.
But this is contrary to the hypothesis that they are ∥.
∴ PQ, XY cannot be produced to meet :
i.e. they are ∥.

Proposition 17.

THEOREM—*If two straight lines are cut by parallel planes, the segments of the one are in the same ratio as the segments of the other—those segments intercepted between the same planes being homologous terms in the ratios.*



Let the st′ lines AXB, CYD, be cut by the ∥ pl′s PQ, RS, UV, in the p′ts A, X, B and C, Y, D, respectively.

Join AD, cutting pl′ RS in Z; and join ZX, ZY.

Then ∵ the ∥ pl′s RS, UV are cut by the pl′ XZDB,

∴ their lines of section XZ, BD are ∥.

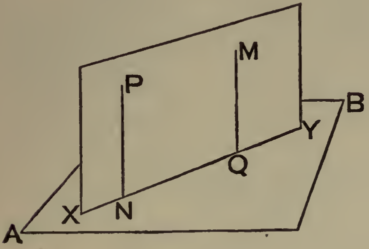
Sim′ly ZY, AC are ∥.

∴ AX : XB = AZ : ZD = CY : YD.

Def′—When two planes intersect so that any line drawn in one of the planes, perpendicular to their line of section, is perpendicular to the other plane, the first plane is said to be **perpendicular** (or **at right angles**) to the second plane.

Proposition 18.

THEOREM—*If a straight line is perpendicular to a plane, then every plane which contains the line is perpendicular to the plane.*



Let st′ line PN be ⊥ to pl′ AB at N.

Take any pl′ cont′g PN; and let its line of section with AB be XY.

Take any p′t Q in XY; and draw, in the pl′ PNX, QM ⊥ to XY.

Then PN, QM, being in same pl′, and each ⊥ to XY, are ∥.

And PN is ⊥ to pl′ AB.

∴ also QM „ „

And ∵ QM, drawn in the pl′ cont′g PN, ⊥ to XY the common section of that pl′ with the pl′ AB, is also ⊥ to the pl′ AB,

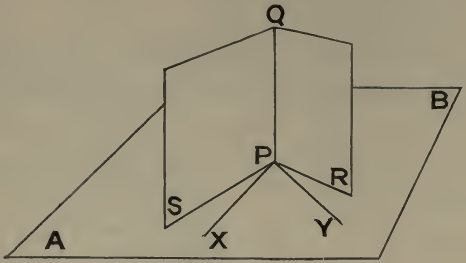
∴ the pl′ cont′g PN is ⊥ to the pl′ AB.

Note—If in plane AB we draw NR perpendicular to XY, then clearly NR is perpendicular to plane PQ.

Hence—*If one plane is perpendicular to a second, then the second is perpendicular to the first.*

Proposition 19.

THEOREM—*If two intersecting planes are each perpendicular to a third plane, their line of section is also perpendicular to that third plane.*



Let pl′s PQR, PQS, which cut in PQ, be each ⊥ to pl′ AB—PR, PS being their respective lines of section with AB.

In AB draw PX, PY respectively ⊥ to PR, PS.

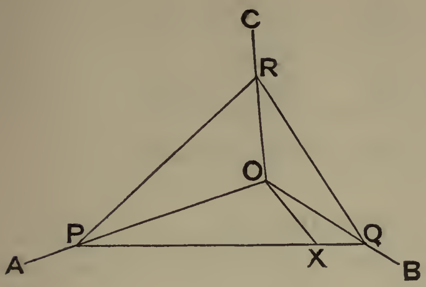
Then PX, being ⊥ to pl′ QR, is ⊥ to PQ;
and PY, ,, ,, QS, ,,
∴ PQ is ⊥ to pl′ PXY: i.e. to pl′ AB.

Def′—When three or more planes meet at a point, then, having regard only to those portions of them which separate a part of space from the rest, they are said to form a **solid** (or **polyhedral**) **angle** at that point; the lines of intersection of the planes are called the *edges* of the solid angle; the plane angles formed by the edges are called the *faces* (or *sides*) of the solid angle; and the point common to the planes is called the *vertex* (or *corner*) of the solid angle.

Note—When a solid angle is formed by three planes it is called a *trihedral* angle.

Proposition 20.

THEOREM—*Any two of the angles which form a trihedral angle are together greater than the third.*



Let the solid angle at O be formed by the three pl′ $\wedge^s$, AOB, BOC, COA.

1°, if the three $\wedge^s$ are equal, of course any two are together greater than the third.

2°, assume that $A\hat{O}B$, $\nless$ $B\hat{O}C$, or $C\hat{O}A$.

At p′t O, in pl′ AOB, draw a finite line OX,

so that $A\hat{O}X = C\hat{O}A$.

Take R in OC, so that $OR = OX$.

Draw any line thro′ X, in pl′ AOB, meeting AO, BO in P, Q respectively; and join PR, QR.

Then in $\triangle^s$ POR, POX,

$$\left.\begin{array}{r} OR = OX, \\ PO \text{ is common,} \\ \text{and } P\hat{O}R = P\hat{O}X\,; \end{array}\right\}$$

$$\therefore \quad PR = PX.$$

$$\text{But } PR + RQ > PX + XQ.$$

$$\therefore \quad RQ > XQ.$$

Hence in $\triangle^s$ QOR, QOX,

$$\left.\begin{array}{r} OR = OX, \\ OQ \text{ is common,} \\ \text{but } RQ > XQ\,; \end{array}\right\}$$

$$\therefore \quad Q\hat{O}R > Q\hat{O}X.$$

$$\therefore \quad Q\hat{O}R + R\hat{O}P > Q\hat{O}X + X\hat{O}P:$$

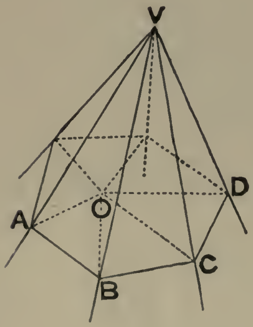
$$\text{i.e.} > P\hat{O}Q.$$

And, as $P\hat{O}Q, \not< R\hat{O}P$, or $Q\hat{O}R$,

$\therefore$ $P\hat{O}Q$ with either of them $>$ the other.

Proposition 21.

THEOREM—*All the plane angles, which form a solid angle, are together less than four right angles.*



Let V be the vertex of a solid $\wedge$, contained by any number of pl′ $\wedge^s$.

Take a pl′ of section of the pl′s cont′g the solid $\wedge$ at V, making a polygon ABCD &c., having as many sides as there are pl′ $\wedge^s$ at V.

Join any p′t O, within this pol′, and in its own pl′, to its corners.

Then sum of $\wedge^s$ of $\triangle VAB$ = sum of $\wedge^s$ of $\triangle OAB$.

Sim′ly for each pair of $\triangle^s$ having a side of pol′ in common.

$$\therefore \quad \wedge^s \text{ at } V + \Sigma(V\hat{B}A + V\hat{B}C) = \Sigma(A\hat{B}C) + \wedge^s \text{ at } O.$$

$$\text{But } V\hat{B}A + V\hat{B}C > A\hat{B}C.$$

$$\therefore \quad \text{sum of } \wedge^s \text{ at } V < \wedge^s \text{ round } O;$$

$$\text{i.e.} < \text{four r′t } \wedge^s.$$

ADDENDA—BOOK xi.

Def—Of all lines which can be drawn from a point to a plane that which is perpendicular to the plane is called the **normal**; and all others are called **obliques.**

Def—If from the ends of a finite line normals are drawn to a plane, the join of their feet is called the **projection** of the line on the plane.

Def—The inclination of a line to a plane is considered to be the angle between the line, produced to meet the plane (if necessary) and its projection on the plane.

THEOREM (1)—*Of all lines which can be drawn from a point to a plane—*

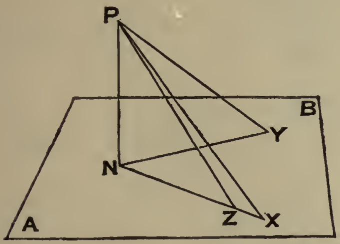
1°, *the normal is minimum:*

2°, *obliques which are equally inclined to the plane are equal; and conversely:*

3°, *obliques whose projections are equal are themselves equal; and conversely:*

4°, *of two obliques, not equally inclined to the plane, that which has the greater inclination to the plane is the shorter; and conversely:*

5°, *of the obliques, whose projections on the plane are unequal, that whose projection is greater is the longer; and conversely.*



Let AB be a pl' to which from a p't P the normal PN is drawn.

1°, $PN <$ any oblique PX, $\because$ $P\hat{N}X$ (being r't) $> P\hat{X}N$.

2°, if PX, PY are obliques making equal $\wedge^s$ with AB;
then in the $\triangle^s$ PNX, PNY, we have

$$\left.\begin{aligned} P\hat{X}N &= P\hat{Y}N, \\ P\hat{N}X &= P\hat{N}Y, \\ \text{and}\quad PN &\text{ common}; \end{aligned}\right\}$$

$$\therefore\quad PX = PY.$$

For the converse: if $PX = PY$; then, as also PN is common to the r't angled $\triangle^s$ PNX, PNY,

$$\therefore\quad P\hat{X}N = P\hat{Y}N.$$

3°, if PX, PY are obliques whose projec's NX, NY are equal;
then in $\triangle^s$ PNX, PNY, we have

$$\left.\begin{aligned} NX &= NY, \\ PN &\text{ common}, \\ \text{and}\quad P\hat{N}X &= P\hat{N}Y; \end{aligned}\right\}$$

$$\therefore\quad PX = PY.$$

So also the converse follows, as in 2°.

4°, if the oblique PY makes a greater $\wedge$ with the pl′ AB than PX, draw the oblique PZ, in pl′ PNX, so that $P\hat{Z}N = P\hat{Y}N$.

Then, by 2°, $PZ = PY$;

and (*E*·*R*·p. 56) $PZ < PX$.

$$\therefore\quad PY < PX.$$

For the converse: if $PY < PX$, draw the oblique PZ, in the pl′ PNX, and equal to PY.

Then (*E*·*R*·p. 56) $P\hat{Z}N > P\hat{X}N$;

and, by 2°, $P\hat{Z}N = P\hat{Y}N$.

$$\therefore\quad P\hat{Y}N > P\hat{X}N.$$

5°, if NX (the projec′ of an oblique PX) $> NY$ (the projec′ of an oblique PY) in NX take Z, so that $NZ = NY$.

Then, by 3°, $PZ = PY$;

and (*E*·*R*·p. 56) $PX > PZ$.

$$\therefore\quad PX > PY.$$

For the converse: if $PX > PY$, draw the oblique PZ, in the pl′ PNX, and equal to PY.

Then (*E·R·*p. 56) $NZ < NX$;

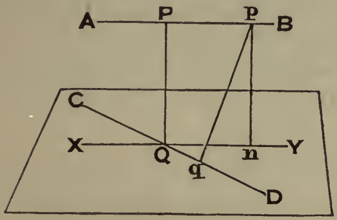
and, by 3°, $NZ = NY$.

$\therefore$ $NX > NY$.

THEOREM (2)—*If two lines neither intersect, nor are parallel, then—*

1°, *there is one, and only one, line perpendicular to each of them; and*

2°, *this common perpendicular is the shortest distance between the lines.*



Let AB, CD be lines which neither intersect, nor are $\parallel$.

Take the pl′ thro′ CD which is $\parallel$ to AB; and let XY be the projection of AB on this pl′.

Since CD is not $\parallel$ to AB, neither is it $\parallel$ to XY: let CD, XY cut in Q.

Then, 1°, since XY is the Locus of the feet of all $\perp^s$, from pts. in AB, on pl′ of CD, XY,

$\therefore$ Q is the only p′t in which a $\perp$ to CD, from a p′t in AB, can meet CD.

$\therefore$ if QP is drawn $\perp$ to AB, it is the only $\perp$ to both AB, CD.

And, 2°, if pq is any other line, joining any p′t p in AB to any p′t q in CD; then, drawing pn $\perp$ to XY, we have

pq (an oblique from p to pl′ CD, XY) $>$ pn (normal to the same pl′)

i.e. $> PQ$.

$\therefore$ PQ is the *min′* dist′ between AB, CD.

Def′—The inclination of a plane to a plane is called a **dihedral angle**; and is measured by the angle between two lines drawn, from any point in the line of section of the planes, one in each plane, perpendicular to this line of section.

Note (1)—That this angle is a proper measure of a dihedral angle is seen from considering that—

1°, it is invariable, wherever in the line of section its vertex is taken; and that,

2°, if two such angles, taken with respect to a pair of dihedral angles, are equal, the dihedral angles can be superposed so as to coincide.

Note (2)—As this angle is equal to one of the angles between any two normals drawn from a point outside the planes to the planes, we may also say that the inclination of a plane to a plane is the angle between two normals to the planes.

Note (3)—It is evident that—

1°, if two planes cut each other, the vertically opposite dihedral angles are equal:

2°, if the planes forming one dihedral angle are respectively parallel to those forming another, the dihedral angles are either equal or supplementary:

3°, if perpendiculars are dropped from any point on two intersecting planes the angle between the perpendiculars is equal to the alternate dihedral angle between the planes:

4°, if a plane cuts two parallel planes it makes—

(*a*) the alternate dihedral angles equal;

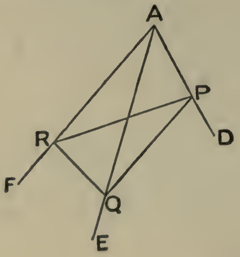
(*b*) the exterior dihedral angle equal to the interior opposite dihedral angle on the same side;

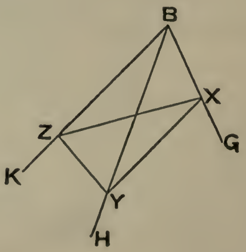
(*c*) the two interior dihedral angles, on the same side, together equal to two right angles.

Theorem (3)—*If two trihedral angles have the three plane angles of the one equal to the three plane angles, similarly situated, of the other, each to each, then—*

1°, *each dihedral angle of the one is equal to the corresponding dihedral angle of the other; and*

2°, *the trihedral angles are identically equal.*





Let the trihedral $\wedge^s$ at A, B, have the three pl′ $\wedge^s$, DAE, EAF, FAD, of the one, respectively equal to the three pl′ $\wedge^s$ GBH, HBK, KBG, of the other; so that the edges AD, AE, AF correspond to the edges BG, BH, BK respectively.

1°, in AD take any p′t P; and in the pl′s DAE, DAF draw PQ, PR respectively, each $\perp$ to AD, to meet the edges AE, AF: join QR.

Take X in BG, so that $BX = AP$; and in the pl′s GBH, GBK draw XY, XZ respectively, each $\perp$ to BG, to meet the edges BH, BK: join YZ.

Then in the Δ^s PAQ, XBY, since

$$\left.\begin{aligned} PA &= XB, \\ \widehat{PAQ} &= \widehat{XBY}, \\ \text{and } \widehat{APQ} &= \widehat{BXY}; \end{aligned}\right\}$$

$$\therefore\ PQ = XY, \quad \text{and} \quad AQ = BY.$$

Again in Δ^s PAR, XBZ, since

$$\left.\begin{aligned} PA &= XB, \\ \widehat{PAR} &= \widehat{XBZ}, \\ \text{and } \widehat{APR} &= \widehat{BXZ}, \end{aligned}\right\}$$

$$\therefore\ PR = XZ, \quad \text{and} \quad AR = BZ.$$

Also in Δ^s QAR, YBZ, since

$$\left.\begin{aligned} AQ &= BY, \\ AR &= BZ, \\ \text{and } \widehat{QAR} &= \widehat{YBZ}; \end{aligned}\right\}$$

$$\therefore\ QR = YZ.$$

Finally, in Δ^s PQR, XYZ, since

$$\left.\begin{aligned} PQ &= XY, \\ PR &= XZ, \\ \text{and } QR &= YZ; \end{aligned}\right\}$$

$$\therefore\ \widehat{QPR} = \widehat{YXZ}:$$

i. e. the dihedral $\wedge^s$ at the edges AD, BG are equal.

Similarly for those at edges AE, BH and those at edges AF, BK.

2°, let the solid $\wedge$ at A be applied to that at B, so that

A coincides with B, edge AD is along edge BG,

and $\widehat{DAE}$ coincides with the equal $\widehat{GBH}$.

Then the pl′ DAF will coincide with the pl′ GBK,

$\because$ the dihedral $\wedge^s$ at edges AD, BG are equal.

$\therefore$ the edge AF will lie along the edge BK,

$\because$ $D\hat{A}F = G\hat{B}K$.

And as also edge AE is along edge BH,

$\therefore$ pl′ EAF coincides with pl′ HBK.

$\therefore$ solid $\hat{A}$ (DEF) $\equiv$ solid $\hat{B}$ (GHK).

Note—Similarly it could be shown that two trihedral angles are identically equal, when they have—

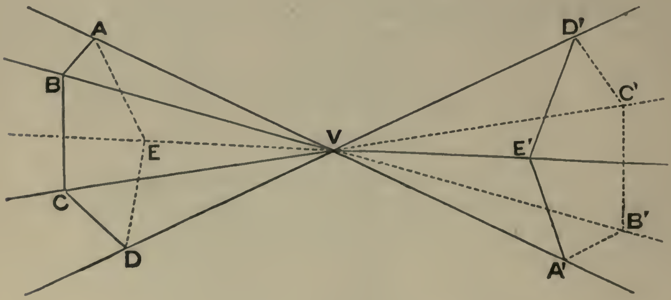
1°, the three dihedral angles of the one equal to the similarly situated dihedral angles of the other, each to each; or

2°, a plane angle and pair of adjacent dihedral angles of the one respectively equal to a plane angle and similarly situated pair of adjacent dihedral angles of the other; or

3°, a dihedral angle and pair of adjacent plane angles respectively equal to a dihedral angle and similarly situated pair of adjacent plane angles of the other.

So also two polyhedral angles which have their plane and similarly situated dihedral angles equal, each to each, taken successively *the same way round*, can be shown to be superposable; and are therefore **identically equal.** But if the equalities only subsist when they are taken *in opposite ways round*, the solid angles are not superposable, though they are, in a sense, equivalent; and they are then said to be **symmetrically equal.**

The meaning of the phrase *taken the same way round*, will appear by considering that the symmetrically equal angle to any given polyhedral angle is constructed by producing each edge of the solid angle through the vertex.



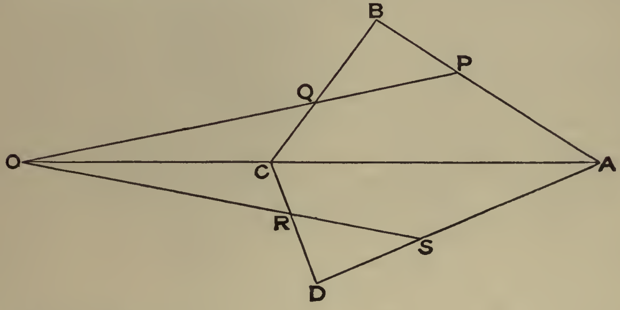
Thus if the edges AV, BV, &c., of the solid angle V (ABCDE) are produced through V in respective directions VA′, VB′, &c., then the plane and dihedral angles, taken in the order ABCDE, are respectively equal to the plane and dihedral angles in the order A′ B′ C′ D′ E′.

But, if an eye is supposed to be looking into the hollow of the first angle, ABCDE is clock-wise; while to an eye looking into the hollow of the second angle A′ B′ C′ D′ E′ is anti-clock-wise.

Def′—When the corners of a quadrilateral (or polygon) are *not* co-planar, it is said to be **gauche.**

THEOREM (4)—*If a gauche quadrilateral is cut by a plane, the ratio compounded of the ratios of the segments of the sides, taken in order one way round, is unity; and conversely.*

Let P, Q, R, S be the p′ts in which a pl′ cuts the respective sides AB, BC, CD, DA of a *gauche* quad′ ABCD.



Let AC meet the pl′ in O.

Then O, Q, P are in pl′ ABC, and also in cutting pl′.

∴ OQP is a st′ line.

Sim′ly ORS ,, .

Then applying *Menelaus*' *Theorem* (*E* · *R* · p. 321) to the Δ^s ACB, ACD,

we get (AP : PB) (BQ : QC) (CO : OA) = 1,

and (CR : RD) (DS : SA) (AO : OC) = 1.

Whence, compounding the ratios, and omitting the reciprocal ratios, we have

(AP : PB) (BQ : QC) (CR : RD) (DS : SA) = 1.

For the converse: assuming the last result to hold for the p'ts P, Q, R, S on sides of *gauche* quad' ABCD; let AC meet PQ in O, and SR in O′: then applying *Menelaus*, as before, and using the result assumed, we get

$$CO : OA = CO' : O'A,$$

i. e. O, O′ are the same p't.

$\therefore$ P, Q, R, S are co-planar.

Note (1)—This Theorem can be generalized for any *gauche* polygon.

Note (2)—A *complete* quadrilateral cannot be *gauche*.

EXERCISES ON PLANES AND SOLID ANGLES.

THEOREMS.

1. If from a point perpendiculars are drawn to a line, and to any plane through the line, the join of their feet is perpendicular to the line.

2. If from P the perpendicular PQ is dropped on a plane, and from Q the perpendicular QN is drawn to any line AB in the plane; then PN is perpendicular to AB.

3. If a trihedral angle is cut by a plane so that the three intercepted edges are equal, the foot of the perpendicular from the vertex on the plane is the circum-centre of the triangle of section.

4. If two lines in one plane are equally inclined to another plane, then they are equally inclined to the common section of these planes.

5. If O is the orthocentre of a triangle ABC, and OP perpendicular to the plane ABC, then PA is perpendicular to the parallel through A to BC.

6. If one triangle lies within another, the sum of the angles subtended at any point, not in the plane of the triangles, by the sides of the outer triangle, is greater than the sum of the angles subtended by the sides of the inner triangle.

Note—If P *is p't*, XYZ *inner* $\triangle$, ABC *outer; let* XY *meet* BC *in* D*: then, by* xi. 20, $\widehat{ZPD} + \widehat{YPD} > \widehat{YPZ}$; *so that sum of* $\wedge^{s}$ *subtended at* P *by* $\triangle$ DXZ $>$ *sum subtended by* $\triangle$ XYZ*: continue this process.*

7. If from the vertex of a trihedral angle a line is drawn within it, then the angle which this line makes with the edges of the solid angle are less than the sum, but greater than half the sum, of the plane angles which form the solid angle.

8. If from a point P, in one of two intersecting planes, a line is drawn at right angles to the plane in which P is, to meet the other in X; and a line from X at right angles to the plane in which X is, to meet the other in Q; then QP is perpendicular to the line of section of the planes.

9. If from P, Q, perpendiculars PX, QY are dropped on a plane; and the plane through P perpendicular to PQ, meets the first plane in AB; then XY, AB are at right angles.

10. If any number of planes have a common line of section, and PX is the perpendicular from any point P on this line; then the Locus of the feet of perpendiculars from P on the planes is the circle on PX as diameter.

11. Four points are so situated that the distance between any pair is equal to the distance between the other pair: prove that the sum of the angles subtended at any one of these points, by each pair of the others, is equal to two right angles.

12. Of the plane angles, forming a trihedral angle, one is right, and the other two supplementary; if a plane is drawn so as to cut off equal lengths from the edges containing the right angle, then the sum of the squares on the three sides of the triangle of section is double the sum of the squares on the edges of the solid angle cut off.

13. If a line meets three parallels to a fixed plane it is always parallel to another fixed plane.

14. A trihedral angle will be identically equal to its symmetrically equal angle if two of the plane angles of the trihedral angle are equal.

15. If a line is equally inclined to any three lines in a plane, whether it meets them or not, it is perpendicular to that plane.

16. If P is a given plane, O a given point in space, and OA, OB parallel to P; then the intersection of the two planes through O, respectively perpendicular to OA, OB, is at right angles to P.

17. With respect to any trihedral angle those planes have a common line of intersection, which are drawn—

1°, through the edges and the bisectors of the opposite faces; or
2°, through the bisectors of the faces perpendicular to those faces; or
3°, through the edges perpendicular to the opposite faces; or
4°, bisecting the dihedral angles.

18. In 4°, of the preceding Exercise, any plane perpendicular to the common line of intersection of the bisecting planes cuts the three internal bisectors and the three external bisectors of the dihedral angles, in lines which form a triangle and its altitudes.

19. If ABC is a triangle, and P, Q points on opposite sides of its plane such that each side of the triangle subtends the same angle at P that it does at Q, then PQ is perpendicular to, and bisected by, the plane of the triangle.

20. In a trihedral angle the sum of the angles formed by the edges, with the bisectors of the opposite faces, is less than the sum of the angles containing the solid angle.

21. In a *gauche* quadrilateral—

1°, the joins of the mid points of consecutive sides form a parallelogram;

2°, the joins of the mid points of opposite sides, and the joins of the mid points of the diagonals, are concurrent.

22. ABCD is a *gauche* quadrilateral: if X, Y, P, Q are points in AB, CD, BC, DA, respectively, such that

$$AX : XB = DY : YC,$$

$$\text{and}\quad BP : PC = AQ : QD; \quad \text{then—}$$

1°, XY, PQ are co-planar;

2°, if XY, PQ cut in O,

$$XO : OY = BP : PC,$$

$$\text{and}\quad PO : OQ = CY : YD;$$

3°, each of the lines XY, PQ is in a plane parallel to the sides which it does not meet.

23. If X, Y are points in the sides AB, CD of a *gauche* quadrilateral ABCD, such that

$$AX : XB = \lambda\,(DY : YC)$$

where λ is a fixed number,

then XY always meets three fixed lines.

24. If, in a *gauche* polygon a new polygon is formed by joining the mid points of its sides, *in any way*; and a second new polygon is formed from the last similarly; and this process is continued indefinitely—which of course can be done in an infinite variety of ways—the polygons will converge finally to a point, which is the same in whatever way the process is performed.

25. The area of the perspective representation, in a given picture, of a triangle of given area, in a fixed plane, varies as the product of the distances of the corners of the perspective representation from the vanishing line. (*Prof' Sylvester: Educational Times Reprint, vol'* I.)

PROBLEMS.

1. If three lines are con-current but not co-planar, draw from their common point a line to make equal angles with them.

2. Find in how many ways a solid angle can be formed with equilateral triangles and squares.

3. A, B are points, one in each of two intersecting planes; find P, in the line of section of the planes, so that PA + PB is *minimum*.

4. XY is a plane, and A, B points not in the plane; find P, in the plane, so that—

1°, when A, B are on same side of XY,

$$PA + PB \text{ is } \textit{minimum};$$

2°, when A, B are on opposite sides of XY,

$$PA \sim PB \text{ is } \textit{maximum}.$$

*Note—See E·R·*p. 306. *The image of* A *in* XY, *is* A′, *got by dropping* AN ⊥ to XY, *and producing it till* NA′ = NA.

5. From the point of section of two lines draw a line perpendicular to each of them.

6. Find the the Locus of points in space equally distant from two intersecting lines.

7. Find the Locus of points in space such that the difference of the squares on the distances of each of them from two fixed points is equal to a given constant.

8. Given a point and two lines, not co-planar; draw from the point a line to meet the lines.

9. Cut a tetrahedral angle, by a plane, so that the section may be a parallelogram.

10. A point moves so that the sum of its distances from two intersecting planes is constant; find its Locus.

11. A line of constant length moves so that its ends are on two fixed lines in space, find the Locus of its mid point.

12. Given a plane and two points not in it; find the Locus of points the ratio of whose distances from the given points is constant.

13. Find the point P, in one of two intersecting lines, such that the perpendicular from P on the other has a given ratio to the distance of P from a given point not co-planar with the lines.

14. Given three lines not co-planar, draw a fourth so as to meet two of them and be parallel to the third.

15. Two planes intersect in XY; if A is a fixed point in one of the planes, and P any point in XY, find when AP makes the *maximum* angle with the other plane.

16. Make a solid angle equal to a given trihedral angle. (*Euclid* xi. 26.)

Note—*If* V *is vertex of given trihed'* $\hat{V}(ABC)$, *drop* AM ⊥ *to pl'* BVC: *make* $Y\hat{S}Z$ *equal to* $B\hat{V}C$, *and* $Y\hat{S}N$ *equal to* $B\hat{V}M$, *where* N *lies within* $Y\hat{S}Z$, *and* SN = VM: *draw* NX ⊥ *to pl'* YSZ *and make* NX *equal to* MA: *then* $\hat{S}(XYZ)$ *will be* ∧ *wanted.*

17. Bisect a given dihedral angle.

18. Three planes are mutually at right angles: cut them by a plane so that the section may be identically equal to a given triangle.

Note—*The edges and corner of the solid* ∧ *project into the alt's and orthocentre of the* △.

19. Given a plane angle, find the Locus of a point in space such that the sum of the projections, on the sides of the angle, of its join to the vertex of the angle, may be constant.

20. Given a *gauche* quadrilateral, and a line which divides two opposite sides proportionally; find a line perpendicular to this line, so that it will divide the other sides of the quadrilateral proportionally.

CHAPTER II.

POLYHEDRA.

Def'—A **solid** has position, length, breadth, and thickness; and, for geometrical purposes, may be considered as a portion of space completely bounded by surface.

Def'—The whole amount of space contained by a specified solid is called its **volume.**

Note—Solids, like plane figures and solid angles, are identically equal when they can be imagined to be superposed so as to produce complete coincidence. In such imagined superposition the solids must be considered capable of interpenetrating each other: in short we consider them to be, as it were, the ghosts of solids.

Def'—A portion of space completely bounded by planes, is called a **polyhedron**; the bounding planes are called the *faces* of the polyhedron; the lines of intersection of the faces are called the *edges* of the polyhedron; and the ends of the edges are called the *corners* of the polyhedron.

Note—Any corner will be properly termed a *vertex* of the polyhedron, when considered with respect to an opposite face taken as *base;* and then the perpendicular from the vertex on the base is called the *altitude* of the polyhedron with respect to that corner and face.

THEOREM (1)—(*Euler's*) *If a convex polyhedron has* E *edges,* C *corners, and* F *faces, then* $E + 2 = F + C$.

Imagine the polyhedron built up by adding its polygonal faces successively; so that the common edges of each new face and the system of old faces shall be continuous.

Let E_r, C_r be values of E and C when r faces have been put together.

Begin with a face of a sides.

$$\text{Then } E_1 = a, \quad C_1 = a.$$

Now add a face of b sides, this will give $b - 1$ new edges, and $b - 2$ new vertices.

$$\therefore \quad E_2 = a + b - 1, \quad C_2 = a + b - 2.$$

$$\therefore \quad E_2 - C_2 = 1.$$

$$\text{Sim'ly} \quad E_3 - C_3 = 2.$$
$$\&c.$$
$$\text{And} \quad E_r - C_r = r - 1.$$

The last result is true until we get to the final (i.e. closing) face, the addition of which gives no new edge or corner.

∴ , if n additions close the polyhedron, we shall have

$$E_n - C_n = E_{n-1} - C_{n-1} = n - 2.$$
$$\text{But} \quad E_n = E, \; C_n = C, \text{ and } n = F.$$
$$\therefore \quad E + 2 = F + C.$$

Listing's Extensions of Euler's Theorem.

(1)—In an open sheet, formed of polygonal facets, $F + C = E + 1$.

(2)—If all space is divided into R convex regions by interfaces, which have F faces, C corners, and E edges, then $F + C = E + R$.

For take out an interface, which is an open sheet: this diminishes R by a unit; and, by (1), the result is true with this interface if it is true without it.

Thus, taking out all the interfaces in succession, we are left with a polyhedron, for which it is true by *Euler's Theorem.*

(3)—In a polyhedron, with an aperture like a ring, $F + C = E$.

For mark off a polygon of edges round the ring: cut it across by a facet with this perimeter; and draw the two cut faces asunder.

Then we have an ordinary polyhedron with two more faces.

Note—When making deductions from *Euler's Theorem* the following notation is useful. In any polyhedron let

f_3 be the number of triangular faces.
f_4 ,, ,, quadrangular ,,
f_5 ,, ,, pentagonal ,,
&c.
s_3 ,, ,, trihedral angles.
s_4 ,, ,, tetrahedral ,,
s_5 ,, ,, pentahedral ,,
&c.

$$\therefore \quad F = f_3 + f_4 + f_5 + \&c.$$
$$C = s_3 + s_4 + s_5 + \&c.$$

Now each edge belongs to two faces, and ends in two corners.

$$\therefore \quad 2E = 3f_3 + 4f_4 + 5f_5 + \&c.$$
$$\text{and} \quad 2E = 3s_3 + 4s_4 + 5s_5 + \&c.$$

Whence, e.g., $f_3 + f_5 + f_7 + \&c.$ is an *even* number,
so also $s_3 + s_5 + s_7 + \&c.$ „ „

Hence the Theorem: *In any polyhedron—*
1°, *an even number of faces have each an odd number of sides ;*
2°, *an even number of solid angles have each an odd number of edges.*

Def′—a polyhedron whose faces are equal and regular polygons is called regular.

Theorem (2)—*There cannot be more than five regular polyhedra.*

For let n be the number of sides in one of the faces of a reg′r polyhed′; and α the number of degrees in an $\wedge$ of a face.

$$\text{Then} \quad \alpha = \frac{n-2}{n} \cdot 180^\circ \; (E \cdot R \cdot \text{ p. } 57).$$

1°, if $n = 6$, $\alpha = 120^\circ$, and $3\alpha = 360^\circ$.

But sum of pl′ $\wedge^s$ forming a solid $\wedge < 360^\circ$.

$\therefore$ the $\wedge^s$ of reg′r hexagons cannot form a solid $\wedge$.

And, *à fortiori*, no solid $\wedge$ can be formed of reg′r pol′s of more than 6 sides.

2°, if $n = 5$, $\alpha = 108^\circ$, and $3\alpha = 324^\circ$.

$\therefore$ 3 reg′r pentagons will form a solid $\wedge$; but not more than 3.

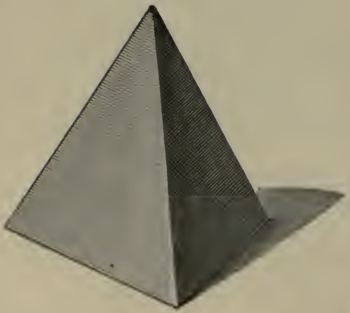
3°, if $n = 4$, $\alpha = 90^\circ$, and $3\alpha = 270^\circ$.

$\therefore$ 3 squares will form a solid $\wedge$; but not more than 3.

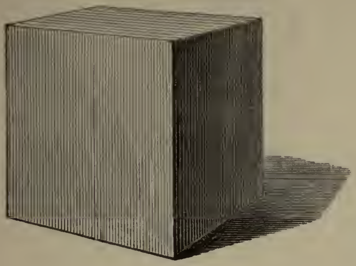
4°, if $n = 3$, $\alpha = 60^\circ$, $3\alpha = 180^\circ$, $4\alpha = 243^\circ$, $5\alpha = 300^\circ$.

$\therefore$ 3, 4, or 5 equilat′ Δ^s, but not more, will form a solid $\wedge$.

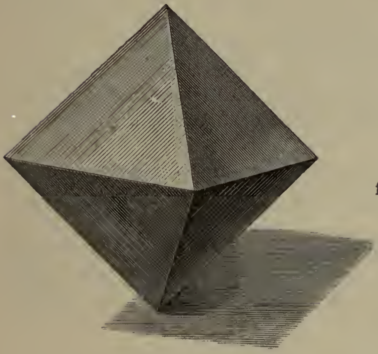
Thus there cannot be *more* than five regular polyhedra: these are—



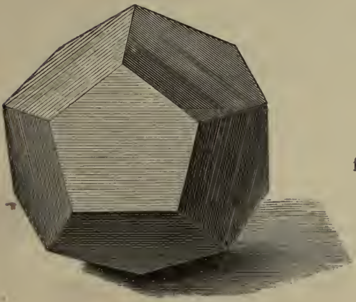
(1) The regular **tetrahedron** (or **triangular pyramid**) formed by four equilat′ Δ^s.



(2) The regular hexahedron (or cube) formed by six sq's.



(3) The regular octahedron formed by eight equilat' Δ^s.



(4) The regular dodecahedron; formed by twelve reg'r pent's.

(5) The regular icosahedron; formed by twenty equilat′ △^{s}.

That each of these has an existence may be proved by solving the Problem of its construction.

The following are outlines of a mode of construction in each case—

1°, for the **tetrahedron**—take an equilat′ △, and erect a ⊥ to its pl′, at its centre: from each of its corners, inflect, to meet this ⊥, lines equal to a side of the △: these lines will, with the sides of the original △, form a regular tetrahedron.

2°, for the **cube**—take a sq′, and at each of its corners erect a ⊥ to its pl′, and make each of them equal to a side of the sq′: then joining their ends successively, a cube is formed.

3°, for the **octahedron**—take a sq′, and, at its centre, erect a ⊥ to its pl′, producing it both ways indefinitely: from the corners of the sq′ inflect lines, each equal to a side of sq′, and drawn on each side of its pl′, to meet this ⊥: the figure formed is a regular octahedron.

4°, for the **dodecahedron**—take a reg′r pent′, and from a corner draw a line equal to its side, and making with each adjacent side an ∧ the same as an ∧ of the pent′: do the same at each of the corners: complete the pent′s thus partially formed: this gives a cup, with a serrated edge, made of six reg′r pent′s. Construct another cup identical with this. The two cups will fit together at their serrated edges, and form a regular dodecahedron.

5°, for the **icosahedron**—take a reg′r pent′, and erect a ⊥ at its centre: inflect lines from its corners to meet this ⊥, so that each of them = a side of pent′: this gives a reg′r pentagonal pyramid (cf. p. 48): then on each side of the original pent′ describe an equilat′ △, so that its pl′ is inclined to the adjacent triangular face of the pyr′ at the same ∧ as any two adjacent triangular faces of the pyr′ are inclined to each other. This gives a cup with a serrated edge made up of ten equilat′ △^{s}. Make another cup identical with the former. The two cups will fit together at their serrated edges, and form a regular icosahedron.

TABLE SHOWING THE NUMBER OF PARTS OF EACH REGULAR SOLID.

NAME OF POLYHEDRON.	*N° of faces.*	*N° of edges.*	*N° of solid angles.*	*N° of sides in each face.*	*N° of plane angles at each corner.*
Tetrahedron ...	4	6	4	3	3
Cube	6	12	8	4	3
Octahedron	8	12	6	3	4
Dodecahedron...	12	30	20	5	3
Icosahedron......	20	30	12	3	5

The discussions of the dodecahedron and icosahedron are tedious, complex, and lead to nothing further: they are, in fact, isolated geometrical curiosities. There is however one striking relation between them worth noticing.

On examination of the table above we see that

N° of solid $\wedge^s$ in tet′ = N° of faces in tet′
,, ,, ,, cube = ,, ,, oct′
,, ,, ,, oct′ = ,, ,, cube
,, ,, ,, dodec′ = ,, ,, icos′
,, ,, ,, icos′ = ,, ,, dodec′.

From this it is evident that the centres of the faces of a reg′r solid are the vertices of an inscribed solid—the solids being related thus—

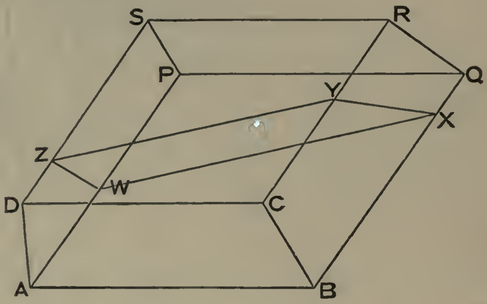
a tet′ may be inscribed in a tet′
an oct′ ,, ,, cube
a cube ,, ,, an oct′
an icos′ ,, ,, a dodec′
a dodec′ ,, ,, an icos′.

Note—The regular polyhedra were formerly called *The Platonic Bodies;* and the consideration of their properties seems to have much occupied the attention of Greek Geometricians. It is noticeable that, though the three simpler forms of them enter largely into Crystallography, no crystal has yet been found of the shape of either of the two more complex.

PARALLELEPIPEDS.

Def′—A polyhedron bounded by six planes, in such a manner that each opposite pair are parallel, is called a **parallelepiped**; and the four joins of its opposite corners (drawn through the solid) are called its *diagonals*.

THEOREM (3)—*The opposite faces of a parallelepiped are identically equal parallelograms; and any section of it by a plane, which cuts two opposite pairs of faces, without cutting the remaining pair, is a parallelogram.*



Let the polyhed′ ABCDPQRS be contained by six pl′s, whereof the pair AC, PR are ∥; as also the pair CS, BP; and the pair BR, AS.

Since the ∥ pl′s AS, BR are cut by pl′ QS,

∴ their intersections PS, QR are ∥.

Sim′ly PQ, RS are ∥.

∴ PQRS is a ▱.

Sim′ly every face is a ▱.

∴ AB = PQ, and BC = QR.

Also, since AB, BC are respectively ∥ to PQ, QR,

$$\therefore\ A\hat{B}C = P\hat{Q}R.$$

∴ ▱ ABCD ≡ ▱ PQRS.

Sim′ly for each other opposite pair of faces.

Now let any pl′ WXYZ cut the opposite pairs of faces AS, BR and BP, CS; but not cut PR, AC.

Then WZ, XY are ||, ∵ they are the sections of the || pl's AS, BR by the pl' WY.

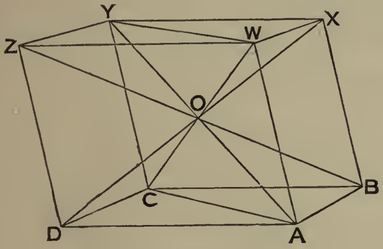
Sim'ly WX, YZ are || ;
∴ WXYZ is a ▱.

Note—If the plane were also to cut the planes PR, AC, the section would be a hexagon.

THEOREM (4)—*The four diagonals of a parallelepiped have*—

1°, *a common mid point ; and*

2°, *the sum of the squares on them equal to the sum of the squares on the twelve edges, or to four times the sum of the squares on any three conterminous edges.*



Let ABCDWXYZ be a $\|^{\text{ped}}$, so that AY, BZ, CW, DX are its diag's.

Then, 1°, AW, CY, being each equal and || to BX, are equal and || to each other.

∴ ACYW is a ▱;

and ∴ its diag's AY, CW, bisect each other.

Sim'ly CW, BZ bisect each other ; as also BZ, DX ; and DX, AY.

∴ there is a p't O which is the mid p't of all four.

And, 2°, $AY^2 + BZ^2 + CW^2 + DX^2$ (*E. R.* p. 107).

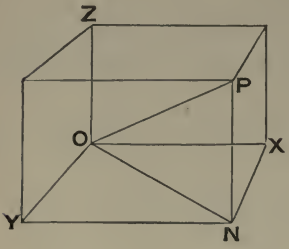
$$= AW^2 + CY^2 + YW^2 + CA^2 + BX^2 + DZ^2 + XZ^2 + BD^2$$

$$= AB^2 + BC^2 + CD^2 + DA^2 + XY^2 + YZ^2 + ZW^2 + WX^2 + AW^2 + BX^2 + CY^2 + DZ^2.$$

$$= 4\,(AB^2 + AD^2 + AW^2).$$

Def'—When the faces of a parallelepiped are rectangles, it is called a **rectangular parallelepiped** ; and when these rectangles are also squares, it is a **cube**.

THEOREM (5)—*The square on a diagonal of a rectangular parallelepiped is equal to the sum of the squares on three adjacent edges.*



Let OX, OY, OZ be three concurrent edges of a rect'r $\|^{\text{ped}}$; OP the diag' thro' O; N the rem'g corner of ▭ XOY.

Then PN is $\perp$ to pl' XOY; so that $\widehat{PNO}$ is r't.

$$\therefore \quad OP^2 = ON^2 + PN^2 = OX^2 + NX^2 + OZ^2 = OX^2 + OY^2 + OZ^2.$$

Cor'—In the case of a cube—

$$(\text{Diag}')^2 = 3\ (\text{an edge})^2.$$

THE METHOD OF LIMITS.

Euclid's treatment of the surfaces and volumes of solids, is extremely tedious; and we shall therefore substitute for it the much more compact, though not less accurate, method of '*Limits*.' The basis of this method is given by Newton (*Principia*, *Book* I. *Lemma* I) in these words—

'*Quantities, and the ratios of quantities, which, in any finite time, tend constantly to equality; and, before the end of that time, approach nearer to each other than by any assignable difference, become ultimately equal.*'

For proof of this Newton says—

'If you deny it; let them become ultimately unequal, and let their ultimate difference be D. Therefore they are unable to approach nearer to equality than by the assignable difference D: contrary to the hypothesis.'

Another way of putting the same idea is this—

If A, B, C, D are quantities such that

$$A : B = C : D;$$

and if P, Q are quantities such that they can be made as near as we please to C, D respectively (where P $>$ or $<$ C, according as Q $>$ or $<$ D),

$$\text{then, ultimately, } A : B = P : Q.$$

For suppose that $C = P + X$, and $D = Q + Y$: where X and Y may be made as small as we please.

$$\text{Then } A : B = P + X : Q + Y.$$

$$\text{Now if } A : B \neq P : Q,$$

let Z be such a quantity that

$$A : B = P : Q + Z.$$

$$\therefore \quad P + X : Q + Y = P : Q + Z.$$

Since, by hypoth′, X, Y can be taken as small as we please, take them so that $Y < Z$.

$$\therefore \quad Q + Y < Q + Z.$$

$$\therefore \quad P + X < P;$$

which is absurd.

Sim′ly the assumption that

$$A : B = P : Q - Z,$$

would lead to an absurdity.

$$\therefore \quad A : B = P : Q.$$

As examples—

Let reg′r pol′s of n sides be inscribed in two $\odot^{s}$ A, B, whose radii are R, r.

Then perim′ of pol′ in A : perim′ of pol′ in $B = R : r$.

But, by increasing n sufficiently, we can make

$$\left.\begin{array}{l}\text{circumf}' \text{ of } A : \text{circumf}' \text{ of } B \\ = \text{perim}' \text{ of pol}' \text{ in } A : \text{perim}' \text{ of pol}' \text{ in } B\end{array}\right\} \text{as near as we please.}$$

$$\therefore \quad \text{circumf}' \text{ of } A : \text{circumf}' \text{ of } B = R : r.$$

Again, area pol′ in A : area pol′ in $B = R^2 : r^2$.

But, by increasing n sufficiently, we can make

$$\left.\begin{array}{l}\text{area of } A : \text{area of } B \\ = \text{area pol}' \text{ in } A : \text{area pol}' \text{ in } B\end{array}\right\} \text{as near as we please.}$$

$$\therefore \quad \text{area of } A : \text{area of } B = R^2 : r^2.$$

Def′—That quantity to which a second quantity continually approaches, and from which it can be made to differ by less than any assignable magnitude, is termed the **Limit** of the second.

E.g. The *Limit* of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c, *ad infin′* is 2.

The *Limit* of the reg′r inscribed and circumscribed pol′s of a ⊙ is the ⊙ itself.

A quantity approaches its Limit by the variation of some other quantity on which it depends: thus in Trigonometries it is shown that the Limit of $\frac{\sin\theta}{\theta}$, as θ diminishes indefinitely, is unity.

This will be indicated by the notation

$$L^{t}_{\theta=0}\left(\frac{\sin\theta}{\theta}\right) = 1.$$

$$\text{So} \quad L^{t}_{n=\infty}\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \&\text{c.} + \frac{1}{2^{n-1}}\right) = 2.$$

$$\text{And} \quad L^{t}_{n=\infty}\left\{\begin{array}{c}\text{Reg'r pol' of } n \text{ sides inscribed in,}\\ \text{or circumscribed about, a } \odot\end{array}\right\} \text{ is the } \odot.$$

Note—By the *def'* of π (*E · R* · p. 26);

$\pi : 1$ = two right angles : a radian,
= semi-circumf' of a ⊙ : an arc of it equal to its radius.
= circumf' : $2r$,
∴ circumf' of a ⊙ = $2\pi r$.

Again, area of a reg'r pol' = $\frac{1}{2}$ rect' under its perim' and its apothem.
But, when the number of sides of the pol' is increased indefinitely,
the Limit of its area = area of its circum-⊙,
" " perim' = circumf' "
" " apothem = radius "
∴, ultimately, area ⊙ = πr^2.

On the Algebraic Representation of Geometric Magnitude.

1°, if two given mag's A, B, are commensurable, there is some unit X, such that

$$A = mX, \text{ and } B = nX;$$

where m, n are whole numbers;
and, in this case, A and B are properly represented by m and n respectively.

2°, if A, B are incommensurable, so that they have no common unit of measurement;

$$\text{let } mA \text{ lie between } nB \text{ and } (n+1)B.$$

$$\therefore \quad A \text{ lies between } \frac{n}{m}B \text{ and } \left(\frac{n}{m} + \frac{1}{m}\right)B.$$

Now, by making m large enough, we can bring these two Limits of A as near together as we please: i.e. as near as we please to $\frac{n}{m}$ B.

So that, if B is the unit, A is represented by $\frac{n}{m}$, to any degree of accuracy.

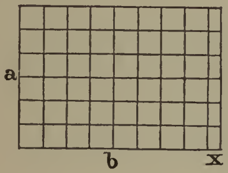
The practical process of finding successively approximate values of $\frac{n}{m}$ is given in Algebra, under the head of '*Continued Fractions.*'

E. g. (as in *E. R.* p. 238) if the side of a sq′ is taken as unit, its diag′ is represented more and more nearly by the fractions

$$\frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \&c.$$

Hence when we represent two geom′ mag′s A, B by letters m and n, we mean that A, B each contain a unit of measurement, either exactly m and n times respectively, or as nearly exactly as we please.

Def′—The **unit of area** is the *square* whose *side* is the *unit of length;* and the **unit of volume** is the *cube* whose *edge* is the *unit of length.*



Let a be the number of units of length in the side of a rect′; then if the other side contains b of the *same* units, the number of units of area in the rect′ is ab.

But if the other side contains this unit b times, with a rem′r cont′g x smaller units, then

area rect′ $=$ ab units of area $+$ a rect′s, each of whose sides are a, x;

i. e. $=$ ab units of area $+$ rect′ under a, x.

But, by taking the original unit small enough, we can make the rect′ under a, x less than any assignable area.

$\therefore$, ultimately, area rect′ $=$ ab units of area.

Now if we suppose a rect′r $\|^{\text{ped}}$, of which this rect′ whose sides are a, b is base, and whose alt′ (i.e. edge $\perp$ to pl′ of rect′) is c; then, if a, b, c are commensurable,

vol′ $\|^{\text{ped}}$ $=$ abc units of vol′.

But, if not, let the alt′ contain the unit of length c times, with a rem′r cont′g y smaller units.

Then vol′ $\|^{\text{ped}}$ $=$ abc units vol′ $+$ a $\|^{\text{ped}}$ whose edges are a, b, y.

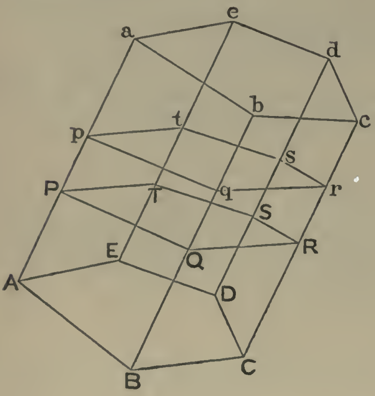
And (as before) by taking the unit small enough, we can make the $\|^{\text{ped}}$ under a, b, y as small as we please.

$\therefore$, ultimately, vol′ $\|^{\text{ped}}$ $=$ abc units vol′.

THE PRISM.

Def′—A **prism** is a polyhedron formed between two parallel, identically equal, and similarly placed polygons, by joining their corresponding corners: these polygons are called the *ends*, or *bases*, of the prism; and the sides formed by joining the corresponding corners—which sides of course are parallelograms—are called the *lateral faces*, or *sides*, of the prism.

THEOREM (6)—*If a prism is cut by parallel planes, the sections are identically equal polygons.*



Let prism, whose ends are ABCDE, abcde, be cut by ∥ pl′s, giving as sections the pol′s PQRST, pqrst.

Then the part of any face (as PQqp) intercepted by the pl′s is a ▭; so that each side of one pol′ (as PQ) = the corresponding side pq of the other.

Also any $\widehat{PQR}$ = the corresponding $\widehat{pqr}$.

∴ pol′ PQRST ≡ pol′ pqrst.

Def′—A section of a prism made by a plane perpendicular to its edges is called a **right section**.

THEOREM (7)—*The lateral surface of a prism is equal to the rectangle under a lateral edge and a line equal in length to the perimeter of a right section.*

In the last fig′ let PQRST be a r′t section;

and let l be length of any one of the lat′ edges.

Then area ▭ ABba = Aa . PQ = l . PQ.

And sim′ly for each of the others.

∴ sum of areas of ▭s (i. e. area lat′ surf′ prism)

= l (PQ + QR + &c.,

i. e. = l . perim′ of r′t section.

Def′—A prism is said to be **right**, or **oblique**, according as its ends are, or are not, perpendicular to its edges.

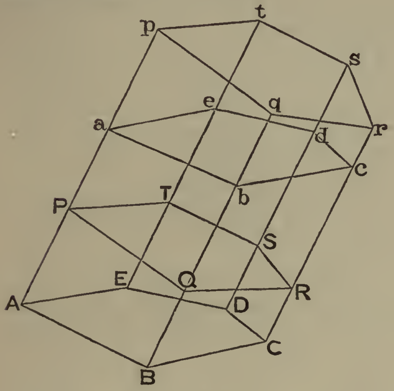
THEOREM (8)—*If* α *is the number of units of length in a line, and* β *the number of units of area in a polygon ; then* $\alpha\beta$ *is the number of units of volume in a right prism of altitude* α *and base* β.

For each unit of area in the base gives a rectangular $\|^{\text{ped}}$, whose base is the unit of area, and altitude α ; and which $\therefore$ contains α units of vol′.

And the whole prism is made up of β such $\|^{\text{peds}}$.

$$\therefore \quad \text{vol}' \text{ prism} = \alpha\beta \text{ units of vol}'.$$

THEOREM (9)—*If* β *is the number of units of area in the right section of an oblique prism, and* α *the number of units of length in a lateral edge ; then its volume is* $\alpha\beta$ *units of volume.*



Let **ABCDE**, **abcde**, be the bases of an oblique prism ; **PQRST** a r′t section of the prism.

Produce **aA** to **p**, so that **ap** = **AP** ; and thro′ **p** let the pl′ ∥ to the r′t section be drawn, cutting the produced edges of the prism in **p**, **q**, **r**, **s**, **t** ; and forming a r′t section of the extended prism.

If we suppose the edges of the prism **pc** to be shifted along their own directions until **q** coincides with **Q**, the prism **pc** will then coincide with the prism **PC** ; ∵ **BQ** = **bq**, **CR** = **cr**, &c. ;

$$\therefore \quad \text{prism } \mathbf{PC} \equiv \text{prism } \mathbf{pc}.$$

$$\therefore \quad \text{vol}' \text{ prism } \mathbf{Ac} = \text{vol}' \text{ right prism } \mathbf{Pr} ;$$

$$\text{i. e.} = \alpha\beta.$$

Cor′ (1)—Prisms on equal bases and of equal alt′s have equal vol′s.

Cor′ (2)—Prisms on equal bases are prop′l to their alt′s.

Cor′ (3)—Prisms of equal alt′s are prop′l to their bases.

Cor′ (4)—As in *E·R·* p. 274, the last two *Cor′s* give—Prisms are to each other in the ratio which is compounded of the ratios of their bases and alt′s.

The Pyramid.

Def′—If any number of triangles are so placed that they have one corner in common, and that their sides opposite this corner form a polygon, then the polyhedron formed by the triangles and the polygon is called a **pyramid**; the polygon is called the *base* of the pyramid; the triangles are called the *lateral faces*, or *sides*, of the pyramid; the common corner of the triangles is called the *vertex* of the pyramid; and the perpendicular from the vertex on the base is called the *altitude* of the pyramid.

Note (1)—When the polygon is regular, and its centre coincides with the foot of the altitude of the pyramid, then the pyramid is said to be *regular*; and the altitude is called its *axis*.

Note (2)—A pyramid is said to be *triangular*, *quadrilateral*, *pentagonal*, &c., according as its base is three, four, five, &c. sided.

Note (3)—When the polygon is a triangle the pyramid is contained by four triangles, and, in that case, any one of these triangles may be considered as base, and any corner as vertex. A triangular pyramid is usually called a *tetrahedron*—the regular form of which has been considered among the five regular solids (p. 36). Among polyhedra the tetrahedron occupies the same position of relative importance that the triangle does among polygons.

Note (4)—Two pyramids are said to be *symmetrical*, when their bases may be made to coincide, and, being made to coincide, the join of their vertices is then bisected at right angles by the base-plane.

Def′—The part of a pyramid intercepted between two planes parallel to its base, is called a **frustrum of a pyramid**, or a **truncated pyramid**; and the sections made by the planes are called the *ends*, or *bases*, of the frustrum.

Theorem (10)—*If* p *is the perimeter of the base of a regular pyramid,* p' *the perimeter of a section parallel to the base,* h *the altitude (drawn from the vertex of the pyramid) of one of its lateral faces, and* a *the distance between a corresponding pair of sides of the ends of the frustrum; then—*

1°, *the lateral surface of the pyramid is* $\frac{1}{2}\,\mathrm{hp}$; *and,*

2°, *the lateral surface of the frustrum is* $\frac{1}{2}\,\mathrm{a}\,(\mathrm{p}+\mathrm{p}')$.

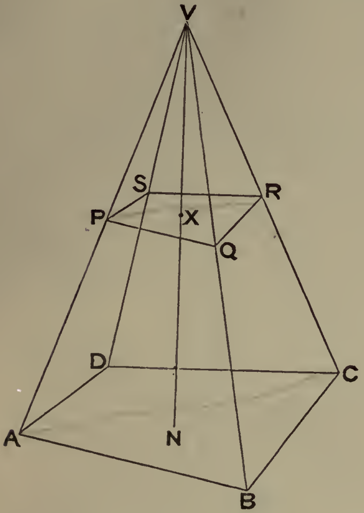
For, 1°, each lat′ face of pyr′ is a $\triangle$ whose area is $\frac{1}{2}\,\mathrm{h}$ (its base).

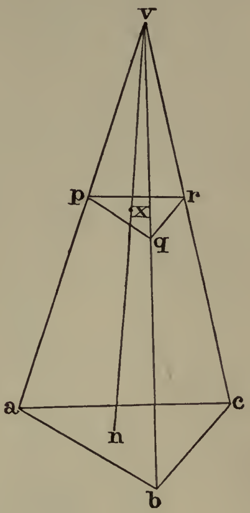
$$\therefore \text{lat}' \text{ surf}' \text{ of pyr}' = \tfrac{1}{2}\,\mathrm{hp}.$$

And, 2°, each lat′ face of frust′ is a trapezium whose area is $\frac{1}{2}\,\mathrm{a}$ (sum of its unequal sides).

$$\therefore \text{lat}' \text{ surf}' \text{ of frust}' = \tfrac{1}{2}\,\mathrm{a}\,(\mathrm{p}+\mathrm{p}').$$

THEOREM (11)—*If pyramids of equal altitudes are cut by planes parallel to their bases, and at equal distances from their vertices, the sections are in the same ratio as the corresponding bases.*





Let VABCD, **vabc** be pyr's whose alt's VN, **vn** are equal.

Place them so that their bases are in one pl′ and V, **v** on same side of this pl′.

Let a pl′ ∥ to the base-pl′ cut VA, VB, VC, VD, VN in P, Q, R, S, X; and **va**, **vb**, **vc**, **vn** in **p**, **q**, **r**, **x**, respectively.

Then PQ, AB are ∥, ∵ they are sections of ∥ pl's PR, AC by pl′ AQ.

Sim'ly PS, AD are ∥; and so on round the pol's.

$$\therefore\ B\hat{A}D = Q\hat{P}S, \text{ \&c.}$$

$$\text{Also } AB : PQ = VA : VP = AD : PS, \text{ \&c.}$$

∴ ABCD, PQRS are sim'r.

So also **abc**, **pqr** are sim'r.

$$\therefore\ PQRS : ABCD = PQ^2 : AB^2,$$

$$\text{and } pqr : abc = pq^2 : ab^2.$$

$$\text{But } PQ^2 : AB^2 = VX^2 : VN^2 = vx^2 : vn^2 = pq^2 : ab^2.$$

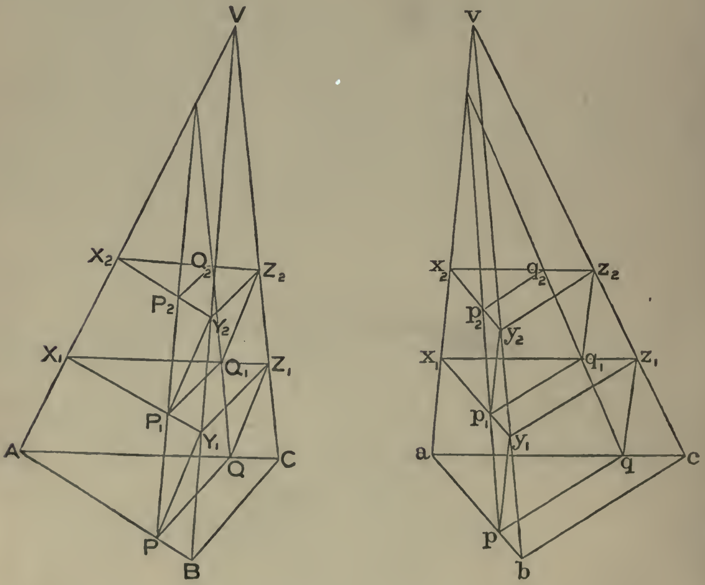
$$\therefore\ PQRS : ABCD = pqr : abc.$$

Cor′ (1)—If pyr′s have the same alt′s and bases of equal area, the sections made by a pl′ $\parallel$ to the base-pl′ are of equal area.

Cor′ (2)—If a pyr′ is cut by a pl′, $\parallel$ to its base-pl′, the lateral edges and alt′ are divided proportionally; and the section is sim′r to the base.

Cor′ (3)—In a pyr′ the areas of sections $\parallel$ to the base are in the dupl′ ratio of their respective dist′s from the vertex.

THEOREM (12)—*If triangular pyramids* (i. e. *tetrahedra*) *have equal altitudes, and bases of equal area, then they have equal volumes.*



Let **VABC**, **vabc**, be pyr′s of equal alt′s, and whose bases **ABC**, **abc**, are of equal area.

Place them so that **ABC**, **abc** are in the same pl′, and **V**, **v** on the same side of that pl′: then **V**, **v** are equidist′ from that pl′.

Divide AV into any number of equal parts in X_1, X_2, &c.

Let pl's thro' X_1, X_2, &c., $\parallel$ to pl' of bases, cut edges VB, VC, &c. in Y_1, Y_2, &c., and Z_1, Z_2, &c. respectively: these pl's will cut the other pyr' in $\triangle^s$ $x_1 y_1 z_1$, $x_2 y_2 z_2$, &c., respectively equal to $\triangle^s$ $X_1 Y_1 Z_1$, $X_2 Y_2 Z_2$, &c.

Thro' $Y_1 Z_1$, $Y_2 Z_2$, &c., draw pl's $\parallel$ to VA, and meeting ABC, $X_1 Y_1 Z_1$, &c., in PQ, $P_1 Q_1$, &c.; and construct for pq, $p_1 q_1$, &c., sim'ly in the other pyr'.

Then vol' prism APQ $X_1 Y_1 Z_1$ = vol' prism apq $x_1 y_1 z_1$,

$\because$ they have equal bases and equal alt's.

Sim'ly vol' prism $X_1 P_1 Q_1 X_2 Y_2 Z_2$ = vol' prism $x_1 p_1 q_1 x_2 y_2 z_2$.

And so on for all the corresponding prisms.

$\therefore$ Σ (prism APQ $X_1 Y_1 Z_1$) = Σ (prism apq $x_1 y_1 z_1$).

Now Σ (prism APQ $X_1 Y_1 Z_1$) differs from pyr' VABC by Σ (prism PBC$Z_1 Y_1$).

And Σ (prism apq $x_1 y_1 z_1$) differs from pyr' vabc by Σ (prism pbc$z_1 y_1$).

$\therefore$ pyr' VABC − pyr' vabc,

= Σ (prism PBC$Z_1 Y_1$) − Σ (prism pbc $z_1 y_1$),

and $\therefore$ $<$ Σ (prism PBC$Z_1 Y_1$),

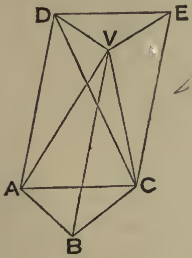
i. e. $<$ frustrum PBCV

$<$ a mag' which can be made as small as we please,

and $\therefore$ = 0, ultimately.

$\therefore$ vol' pyr' VABC = vol' pyr' vabc.

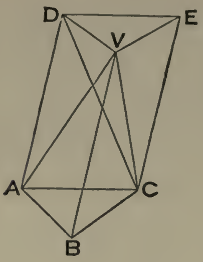
THEOREM (13)—*The volume of a pyramid whose base contains* β *units of area, and whose altitude is* α *units of length, is* $\frac{1}{3}\alpha\beta$ *units of volume.*



1°, take a pyr' VABC, whose base is the $\triangle$ ABC.

Let $\parallel^s$ AD, CE, to BV, meet a pl' thro' V $\parallel$ to ABC in D, E: join DE.

This gives a prism ABC EVD.



Let pl′ DVC cut pl′ ACED in DC.

Then DC bisects the ▭ ACED.

∴ pyr′ VACD = pyr′ VECD,

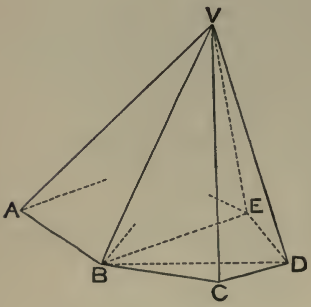
∵ they are on equal bases, and of same alt′.

Also Δ DVE = Δ ABC.

∴ pyr′ CDVE = pyr′ VABC.

∴ pyr′ VABC = $\frac{1}{3}$ prism,

$= \frac{1}{3}\alpha\beta.$



2°, take a pyr′ VABCD, &c., whose base is the polygon ABCD, &c.

The pl′s VBD, VBE, &c., divide it into pyr′s whose bases are Δ^s BCD, BDE, &c.; and all these pyr′s have same alt′ as original pyr′.

$$\therefore \text{ pyr}' \text{ VABC \&c.} = \tfrac{1}{3}\alpha\,(\Delta\text{ BCD} + \Delta\text{ BDE} + \text{\&c.})$$
$$= \tfrac{1}{3}\alpha\beta.$$

Cor′ (1)—Pyr′s on equal bases, and of equal alt′s have equal vol′s.

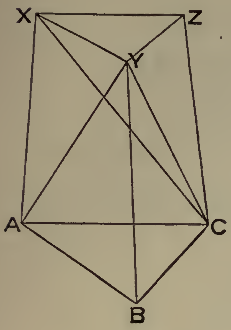
Cor′ (2)—Pyr′s on equal bases are prop′l to their alt′s.

Cor′ (3)—Pyr′s of equal alt′s are prop′l to their bases.

Cor′ (4)—As in *E. R.* p. 274, the last two *Cor′s* give—

Pyr′s are to each other in the ratio which is compounded of the ratios of their bases and alt′s.

THEOREM (14)—*The volume of a frustrum of a pyramid is equal to the sum of the volumes of three pyramids, each of which has the same altitude as the frustrum, and whose respective bases are the ends of the frustrum and a mean proportional between those ends.*



Let **h** be the alt′ of the frustrum.

1°, take a frust′ of a tetrahedron; let ABC, XYZ be its ends.

The pl′s AYC, XYC cut it into three tet′s YABC, CXYZ, YACX.

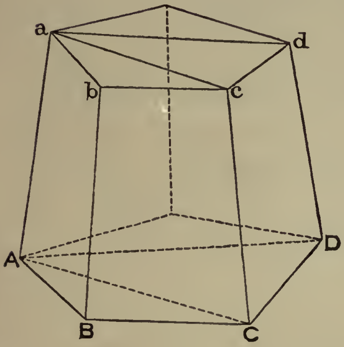
Let V_1, V_2, V_3 be the respective volumes of these.

Then V_1 = pyr′ of alt′ **h** and base ABC;
and V_2 = „ „ XYZ.

Again, taking C as common vertex of pyr′s CAXY, CABY, these have the same alt′.

$\therefore\ V_1 : V_3 = \triangle\,ABY : \triangle\,AXY$
$= AB : XY$, ∵ $\triangle$s have same alt′
$= AC : XZ$, ∵ $\triangle$s ABC, XYZ are sim′r
$= \triangle\,ACX : \triangle\,XZC$
$= V_3 : V_2$,

i.e. V_3 = pyr′ of alt′ **h** and base a mean prop′l between ABC, XYZ.



2°, let the ends of the frust′ be the sim′r and sim′ly placed pol′s ABC, &c., abc, &c.

By joining A and a to each rem′g corner of its own pol′, the pol′s are divided into the same number of sim′r $\triangle$s; also the pl′s such as AC, ca divide the polygonal frust′ into tetrahedral frustra.

∴ vol′ frust′ = sum vol′s of 3 sets of pyr′s having same alt′ $\mathbf{h}$, and such that

bases of one set are Δ^s of pol′ ABC, &c.;

bases of another set are Δ^s of pol′ abc, &c.;

bases of a third set are means between corresponding Δ^s.

The first set = pyr′ of alt′ $\mathbf{h}$ and base the pol′ ABC &c.;
the second set = pyr′ of alt′ $\mathbf{h}$ and base the pol′ abc &c.

For the third set; let P, p be the areas of the pol′s ABC, abc, &c., respectively; and let

Δ_1, Δ_2, &c. be the areas of the successive Δ^s of P;
δ_1, δ_2, &c. ,, ,, ,, p.

Also let M be the mean of P, p;
M_1 ,, Δ_1, δ_1;
M_2 ,, Δ_2, δ_2;
&c.

Then $P : p = \Delta_1 : \delta_1 = \Delta_2 : \delta_2 =$ &c.;
also $P : p =$ dupl′ of $P : M$,
and $\Delta_1 : \delta_1 =$ dupl′ of $\Delta_1 : M_1$,
&c., &c.

∴ (*E*·*R*·v. 9) $P : M = \Delta_1 : M_1 = \Delta_2 : M_2 =$ &c.
$= \Delta_1 + \Delta_2 +$ &c. $: M_1 + M_2 +$ &c.

But $P = \Delta_1 + \Delta_2 +$ &c.;
∴ $M = M_1 + M_2 +$ &c.

∴ the third set of pyr′s = pyr′ of alt′ $\mathbf{h}$ and base M.

∴ the Theorem is universally true.

Note—If the ends (or bases) of the frustrum are denoted by B, b; the foregoing Theorem gives the following formula for arithmetic calculation

$$\text{vol′ of frust′} = \frac{h}{3}(B + b + \sqrt{Bb}).$$

This again may be conveniently modified as follows—

Let K be the ratio of any pair of homol′ sides of the bases, so that

$$b : B = K^2 : 1,$$
$$\therefore\ b = BK^2.$$

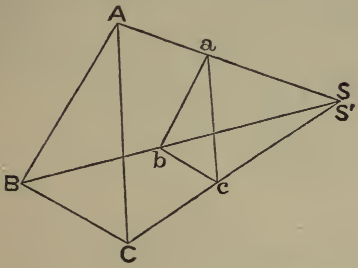
Then above formula becomes

$$\text{vol′ frust′} = \frac{Bh}{3}(1 + K + K^2).$$

Def′—Polyhedra are said to be **similar** when they are contained by faces—
1°, similar and similarly situated;
2°, forming equal similarly situated dihedral angles;
3°, forming equal similarly situated solid angles.

Note—Of the above conditions any two being true necessitates the third.

THEOREM (15)—*If the lesser of two similar polyhedra is placed within the greater, in such a manner that each pair of homologous edges is parallel, then the joins of corresponding corners are concurrent.*



Each pair of faces may be divided into the same number of sim′r and sim′ly sit′d Δs.

Let ABC, abc be a pair of such Δs; and let Aa, Bb meet in S, and Cc, Bb in S′.

Then $\text{BS} : \text{bS} = \text{AB} : \text{ab} = \text{BC} : \text{bc} = \text{CS}' : \text{cS}'$,

∴ S, S′ are the same point.

Sim′ly, by taking the Δs successively all over the polyhedra, we should get concurrency at S, for all joins of corresponding corners.

Def—The point S is called the **centre of similarity** of the two polyhedra.

THEOREM (16)—*Similar polyhedra are in the triplicate ratio of their homologous edges.*

1°, Consider two sim′r pyr′s: they are to each other in the ratio which is compounded of the ratios of their bases and alt′s.

But, taking a pair of r′t angled Δs, formed by the alt′s, a pair of homologous edges, and the joins of the feet of the alt′s to the end of the correponding edge, we get—

alt′ of one : alt′ of other = edge of 1st : edge of 2nd.

Again, the bases being sim′r polygons,
base of one : base of other = dupl′ of side of base of 1st : homol′ side of 2nd.
Lastly, the homol′ faces being sim′r $\triangle$s,
side of base of 1st : homol′ side of 2nd
= adjoining edge of 1st : homol′ edge of 2nd.
∴ base of one pyr′ : base of other = dupl′ of any edge : its homol′ edge.
∴ pyr′s are in ratio compounded of a pair of homol′ edges with the dupl′ of a pair of homol′ edges,
i. e. in the tripl′ of a pair of homol′ edges.

2°, Let two sim′r polyhedra be placed so that their homologous edges are ∥: then the lines from the centre of similarity through the corners divide the polyhedra into the same number of sim′r pyr′s.

But each pair of corresponding pyr′s are in the triplicate ratio of their homol′ sides.

∴ sum pyr′s in one polyhedron (i. e. vol′ of polyhedron),
: ,, other ,, ,, ,, ,,
= tripl′ ratio of a pair of homol′ edges.

THEOREM (17)—*If four lines are proportional, and if similar polyhedra are similarly described on the first pair as edges, and similar polyhedra are similarly described on the second pair as edges, then as the polyhedron on the first is to the polyhedron on the second so is the polyhedron on the third to the polyhedron on the fourth; and, conversely, if two similar polyhedra have the same ratio as two other polyhedra, similar to each other, then similar edges of the first pair have the same ratio as similar edges of the second pair.*

LEMMA—*If two ratios are equal, their triplicates are equal; and conversely.*

Let A, B, C, D, P, Q, X, Y be mag′s such that,

$$A : B = B : X = X : Y,$$
$$C : D = D : P = P : Q.$$

1°, if $A : B = C : D$,
then $B : X = D : P$,
and $X : Y = P : Q$,

whence, compounding the ratios, we get

$$A : Y = C : Q,$$

i. e. triplicate of $A : B$ = triplicate of $C : D$.

2°, if $A : Y = C : Q$,

assume that $A : B = C : \alpha = \alpha : \beta = \beta : \gamma$.

$$\therefore\ A : Y = C : \gamma.$$

$$\therefore\ \gamma = Q. \qquad \text{(1)}$$

Now if $\alpha < D$, then $\beta < P$,

and $\therefore\ \gamma < Q$;

which contradicts (1).

And if $\alpha > D$, then $\beta > P$,

and $\therefore\ \gamma > Q$;

which again contradicts (1).

It remains $\therefore$ that $\alpha = D$.

$$\therefore\ A : B = C : D.$$

Now let AB, CD, LM, NO be four given lines.

On AB, CD let sim'r polyhedra P, Q be sim'ly described, and on LM, NO let sim'r polyhedra X, Y be sim'ly described.

1°, suppose $AB : CD = LM : NO$.

$$\begin{aligned} \text{Then } P : Q &= \text{tripl}' \text{ ratio of } AB \text{ to } CD \\ &= \text{tripl}' \text{ ratio } LM \text{ to } NO \\ &= X : Y. \end{aligned}$$

2°, suppose $P : Q = X : Y$.

$$\begin{aligned} \text{Then tripl}' \text{ ratio of } AB \text{ to } CD &= P : Q \\ &= X : Y \\ &= \text{tripl}' \text{ ratio of } LM \text{ to } NO. \\ \therefore\ AB : CD &= LM : NO. \end{aligned}$$

THE DELIAN PROBLEM—*To make a cube whose volume is double that of a given cube.*

Let α be an edge of the given cube.

Find (*E. R.* p. 276) two mean prop'ls x, y between α and 2α, so that

$$\alpha : x = x : y = y : 2\alpha.$$

Construct a cube on x.

$$\begin{aligned} \text{Then cube on } \alpha : \text{cube on } x &= \text{tripl}' \text{ of } \alpha : x \\ &= \alpha : 2\alpha \\ &= 1 : 2. \\ \therefore\ \text{cube on } x &= 2 \text{ cube on } \alpha. \end{aligned}$$

Cor'—By taking $m\alpha$, in the above, instead of 2α, we should get a cube m times the given cube.

EXERCISES ON POLYHEDRA.

THEOREMS.

1. If the edges of a tetrahedron are all equal the sum of the angles at any corner is equal to two right angles.

2. In a regular tetrahedron three times the square on an altitude is equal to twice the square on an edge.

3. In a regular tetrahedron an altitude is equal to three times the perpendicular from its own foot on a face.

4. If the base of a tetrahedron is an equilateral triangle, and each of the angles at its vertex is right; then—

1°, the square on its altitude is equal to one-third of the square on an edge terminated in the vertex; and

2°, the sum of the perpendiculars on the faces, from any point in the base, is constant.

5. If the opposite edges of a tetrahedron are equal, two and two, the faces are acute angled.

6. The necessary and sufficient condition that the altitudes of a tetrahedron are concurrent is that the sum of the squares on pairs of opposite edges are equal.

7. Two tetrahedra are similar when they have—

1°, a pair of dihedral angles equal, and the containing faces of the one similar and similarly situated to the containing faces of the other, each to each; or

2°, a pair of similar faces, and the adjacent three dihedral angles of the one respectively equal to the similarly situated dihedral angles of the other; or

3°, three faces of the one similar and similarly situated to three faces of the other; or

4°, the five dihedral angles of the one equal to the five similarly situated dihedral angles of the other, each to each.

8. A dihedral angle of a regular tetrahedron is supplementary to a dihedral angle of a regular octahedron.

9. The three rectangles under pairs of opposite edges of a tetrahedron are such that any two of them are together greater than the third.

10. If A, B are opposite corners of a cube; and the mid points of the six edges not through A, B are joined successively; then these joins form a regular hexagon.

11. In any tetrahedron the plane bisecting a dihedral angle cuts the opposite edge into segments proportional to the areas of the adjacent faces.

NOTE—*Project the fig' on a pl' ⊥ to the edge of the dihed' ∧.*

12. Every plane, through the mid points of a pair of opposite edges of any tetrahedron, divides it into parts whose volumes are equal. (*Bobillier's Theorem.*)

NOTE—*Each piece is composed of a pyr' and a tet': the pyr's have pl' of section as a common base; and their alt's are equal: the tet's are in ratio compounded of their bases and alt's; or of the seg'ts of the unbisected sides: then consider the 4 corners of tet' as those of a* gauche *quad' cut by pl' of section.*

13. In a frustrum of a tetrahedron—

1°, the joins of the corners of one base, to the respective opposite mid points of the other base, are concurrent.

2°, the joins of the mid points of the lateral edges to the intersections of the diagonals of the opposite faces are concurrent.

14. If ϕ denote a dihedral angle of a regular solid, prove that the following constructions will give ϕ.

(1) for the tetrahedron: describe a right-angled triangle ABC, in which AB is the hypotenuse, and AC is one-third of AB; then BAC is ϕ.

(2) for the cube, ϕ is a right angle.

(3) for the octahedron: describe a right-angled triangle ABC, in which the sides AC, CB, containing the right angle, are respectively the side and diagonal of a square; then twice BAC is ϕ.

(4) for the dodecahedron: describe a right-angled triangle ABC, in which the sides AC, CB, containing the right angle are to each other as the segments of a line divided in extreme and mean ratio, AC being the lesser; then twice BAC is ϕ.

(5) for the icosahedron: describe a right-angled triangle in ABC, in which the sides AC, CB, containing the right angle are to each other in the duplicate of the ratio of the segments of a line divided in extreme and mean ratio, AC being the lesser; then twice BAC is ϕ.

15. Given the edge of a regular solid, the following construction will give the radii of its in- and circum- spheres. Let ABC be a line such that AB, AC are the radii of the in- and circum- circles of a face; make the angle ABD equal to half the dihedral angle of the solid, so that D is a point in the perpendicular to AB at A: then DA is the radius of the in- sphere and DA of the circum- sphere.

16. The volume of a regular solid is one-third that of a prism whose altitude is the radius of the in-sphere, and base equal to the whole surface of the solid.

17. Deduce from *Euler's Theorem* (p. 34) that—

(1) $E \nless \frac{3}{2}F$, and $E \ngtr 3(F - 2)$.

(2) $C \nless \frac{1}{2}F + 2$, and $C \ngtr 2(F - 2)$.

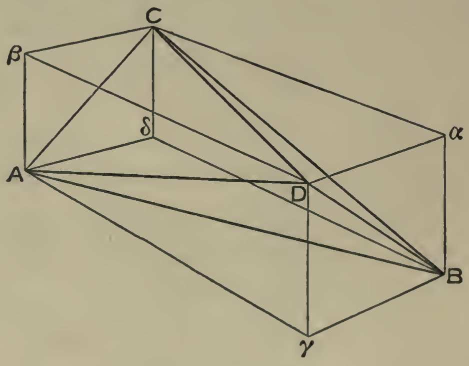
(3) $f_3 + s_3 \nless 8$.

(4) f_3, f_4, f_5 cannot be simultaneously zero; nor can s_3, s_4, s_5.

(5) the sum of the plane $\wedge^s = (C - 2) \times 360^\circ$.

18. The sum of the dihedral angles of a polyhedron, of **n** faces, lies between 2**n** and 2(**n** − 2) dihedral right angles.

19. By means of the following construction* show that the subsequent properties of tetrahedra follow at once.



Let ABCD be a tet'. Thro' each pair of its oppos' edges draw || pl's: this will enclose the tet' in a $\|^{\text{ped}}$, the diag's of whose oppos' faces are oppos' edges of tet'.

Let corners of this solid, oppos' A, B, C, D, be α, β, γ, δ respectively.

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* TORRY—*Messenger of Mathematics*, vol. i, p. 156.

Then—

1°, joins of mid pt′s of oppos′ edges of tet′ go thro′ cent′ of $\|^{\text{ped}}$, and are $\|$ and equal to its edges:

2°, if these joins are $\perp$ to each other, the $\|^{\text{ped}}$ is rectang′r; and then the oppos′ edges of the tet′ are equal, and the faces of the tet′ are identically equal:

3°, if oppos′ edges of tet′ are $\perp$, faces of $\|^{\text{ped}}$ are rhombic:

4°, if 2 pairs of oppos′ edges of tet′ are $\perp$, the 3rd pair will also be $\perp$; for if 2 pairs of oppos′ faces of $\|^{\text{ped}}$ are rhombic, so also is the 3rd pair:

5°, vol′s of pieces cut off from $\|^{\text{ped}}$ by faces of tet′ are each $\frac{1}{6}$ vol′ $\|^{\text{ped}}$;

$$\text{for } \mathsf{DAB\gamma} = \tfrac{1}{2}\,\mathsf{DA\delta B\gamma} = \tfrac{1}{6}\text{ vol}'\ \|^{\text{ped}};$$

and sim′ly for others;

$$\text{so that vol}'\text{ tet}' = \tfrac{1}{3}\text{ vol}'\ \|^{\text{ped}}:$$

6°, mean centre of tet′ coincides with mean centre of $\|^{\text{ped}}$ (cf. p. 73).

7°, sum of sq′s on edges of tet′ = 4 sum of sq′s on joins of mid pt′s of oppos′ edges:

8°, the 4 pl′s each of which contains one edge, and bisects the oppos′ edge of tet′, have one common p′t:

9°, every section of a tet′, by a pl′ $\|$ to a pair of oppos′ edges, is a $\square$:

10°, in a reg′r tet′ the joins of the mid p′ts of adjacent sides form a sq′:

11°, in a reg′r tet′ the *min′* dist′ between 2 oppos′ edges = $\frac{1}{2}$ diag′ of the sq′ on an edge:

12°, in a reg′r tet′ the 3 *min′* dist′s and the 4 alt′s are all concurrent.

Def′—A tetrahedron is said to be **orthogonal** when each pair of its opposite edges is at right angles. By 4°, above, it will be sufficient to constitute a tetrahedron orthogonal that *two* pairs of opposite edges are at right angles.

20. The following are properties of an orthogonal tetrahedron—

NOTE.—*Some terms, not yet defined, are used in the following group: their meanings will be found by referring to the Index.*

(1) The four spheres, each circumscribing one face, and having its centre at a distance from that face equal to half the distance of the fourth vertex from it, are equal.

(2) The sum of the squares on any two opposite edges is equal to the square on the diameter of either of these spheres.

(3) The corners of any face and the orthocentre of the opposite face are co-spheric; and the four such spheres are equal.

(4) The N.P. circles of the four faces are co-spheric.

(5) The radius of the last sphere is half that of either of the spheres in (1) and (3).

(6) An orthogonal tetrahedron (but no other) has an orthocentre.

(7) The six mid points of the edges are on a sphere whose centre is the mean centre of the tetrahedron.

(8) The orthocentre, mean centre, and circum-centre are collinear.

(9) The six points where the minimum distances between opposite edges meet those edges, are co-spheric with the six points in (7).

(10) The sphere through the centroids of the four faces, go through the orthocentres of those faces, and through the points of trisection nearest O (the orthocentre of the tetrahedron) of the joins of O to the four corners.

NOTE.—*This sphere, by analogy with the* N.P. ⊙ *of a* △, *may be properly called the* **twelve-point-sphere** *of the tetrahedron.*

(11) The radius of the *twelve-point-sphere* is one-third that of the circum-sphere; its centre is on the line defined by (8); and the mean centre and orthocentre of the tetrahedron are the internal and external centres of similitude of the twelve-point- and circum- spheres.

(12) The orthocentre of the tetrahedron is the orthocentre of the four triangles in which the tetrahedron is cut by planes through the edges perpendicular to the opposite edges.

(13) When the orthocentre of the tetrahedron is outside the circum-sphere, there is a sphere with respect to which the tetrahedron is self-conjugate.

(14) The radius ρ of this last sphere is given by

$$3\rho^2 = SO^2 - R^2,$$

where S is circum-centre, R radius circum-sphere.

(15) The radius ρ' of sphere in (7) is given by

$$3\rho'^2 = R^2 - \tfrac{1}{4} SO^2.$$

(16) Spheres in (7) and (13) cut orthogonally.

(17) Twelve-point-sphere, circum-sphere, and spheres in (7) and (13) have a common radical plane.

(18) The N.P. circle of any face, and the ortho-centres of the three remaining faces are co-spheric; and the sphere on which they lie bisects the joins of the ortho-centre of the tetrahedron to the corners of the first face; also the four such spheres are equal, and equal to the sphere through the mid points of the edges.*

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* Of this group the first six were given by Mr. T. C. LEWIS in the *Messenger of Mathematics, vol'* XI, p. 36; and the rest by Mr. E. TEMPERLEY on p. 114 of the same vol'.

PROBLEMS.

1. Find the point within a tetrahedron for which the sum of the squares on its joins to the four corners is *minimum*. Show also that, for the same point, the product of perpendiculars from it on the four faces is *maximum*.

2. Through a point inside a trihedral angle draw a plane so that the point shall be the mean centre of the intercepted triangle. (Cf. p. 71.)

3. Find the Locus of a point such that the sum of the pyramids subtended at it by two triangles (not co-planar) is constant.

4. Given the total area of the bounding surface of a rectangular parallelepiped, find when its volume is *maximum*.

5. Cut a cube so that the section may be a regular hexagon.

6. Cut a tetrahedron, by a plane parallel to a pair of opposite edges so that the section may be *maximum*.

NOTE—*The pl' must be drawn thro' the mid p'ts of the other edges.*

7. Cut a regular tetrahedron, by a plane, so that the section may be a square.

8. Find the point within a tetrahedron such that if it is joined to the four corners the tetrahedron is divided into four tetrahedra of equal volume.

9. Bisect a pyramid, whose base is a parallelogram, by a plane through one of the sides of the base.

NOTE—*Draw the pl' thro' the p'ts in which the edges not conterminous with the side of the base, are divided in extreme and mean ratio.*

10. Given the edges of a tetrahedron, construct a line equal to one of its altitudes.

NOTE—*If* V *is vertex*, ABC *base ; imagine each of faces* VAB, VBC, VCA *turned round its edge of the base, until in pl' of base,* V *taking positions* V_1, V_2, V_3: *then* $\perp^s$ *from* V_1, V_2, V_3, *on* AB, BC, CA *are concurrent in foot of alt'; and* $AV_3 = AV$*: construction now evident.*

11. Cut a cube so as to give a regular tetrahedron.

CHAPTER III.

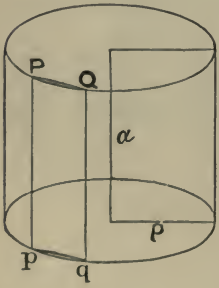
SOLIDS OF REVOLUTION.

THE CYLINDER.

Def—If a rectangle revolves round one of its sides as a hinge, until it returns to its original position, the solid generated is called a **cylinder of revolution**, or a **right circular cylinder**; the side about which it revolves is called the *axis* of the cylinder; either of the revolving sides is called a *radius* of the cylinder; and the circles generated by these radii are called the *ends*, or *bases*, of the cylinder.

Note—Cylindrical surfaces in general are generated by a straight line moving parallel to a fixed direction, and so as always to pass through a fixed curve. In what follows, by cylinder we mean always cylinder of revolution, unless the contrary is stated.

THEOREM (1)—*The curved surface of a cylinder (axis* α, *and radius* ρ) *is* $2\pi\rho\alpha$; *and its volume is* $\pi\rho^2\alpha$.



Let PQ be a side of the reg′r pol′ inscribed in one of the ends of the cyl′.

Let a pl′ thro′ PQ, ∥ to α, meet the other end in pq; and let sim′r pl′s be drawn thro′ each side of the pol′, thus forming a reg′r inscribed polygonal prism of which PpqQ is a lateral face.

Then the cyl′ is the Limit of this prism, when the number of sides of the pol′ is increased indefinitely.

$$\text{Now lat}' \text{ surf}' \text{ of prism} = \alpha \,.\, \text{perim}' \text{ pol}';$$
$$\text{and vol}' \text{ of prism} = \alpha \,.\, \text{area pol}'.$$

∴, passing to the Limit, we get

$$\text{curved surf}' \text{ of cyl}' = 2\pi\rho\alpha;$$
$$\text{and vol}' \text{ of cyl}' = \pi\rho^2\alpha.$$

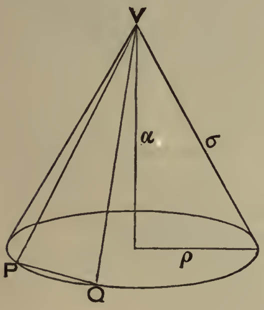
Cor′—Total surf′ of cyl′ $= 2\pi\rho\alpha + 2\pi\rho^2 = 2\pi\rho(\rho + \alpha)$.

THE CONE.

Def′—If a right-angled triangle revolves round one of its sides, containing the right angle, as a hinge, until it returns to its original position, the solid generated is called a **cone of revolution**, or a **right circular cone**; the fixed side is called the *axis* of the cone, the revolving side is called the *radius* of the cone; the end of the axis, not at the right angle, is called the *vertex* of the cone; the hypotenuse of the triangle in any position is called a *slant side* of the cone; and the circle generated by the remaining side is called the *base* of the cone.

Note—Conical surfaces in general are generated by a straight line which revolves so as always to go through a fixed point and a fixed curve. In what follows, by cone we mean always cone of revolution, unless the contrary is stated.

THEOREM (2)—*The curved surface of a cone (axis* α, *slant side* σ, *and radius* ρ), *is* $\pi\rho\sigma$; *and its volume is* $\frac{1}{3}\pi\rho^2\alpha$.



Let PQ be a side of a reg′r pol′ inscribed in the base of the cone.

Join the vertex V to P and Q; and let the same construction be continued round the cone, thus forming a reg′r inscribed polygonal pyr′ of which the pol′ is base, and VPQ one of the lat′ faces.

Then the cone is the Limit of this pyr′ when the number of sides of the pol′ is indefinitely increased.

Now lat′ surf′ of pyr′ $= \Sigma(\triangle\, \mathsf{VPQ}) = \frac{1}{2}\sigma$. perim′ pol′;

and vol′ of pyr′ $= \frac{1}{3}\alpha$. area pol′.

∴, passing to the Limit, we get

curved surf′ of cone $= \frac{1}{2}\sigma \,.\, 2\pi\rho = \pi\rho\sigma$;

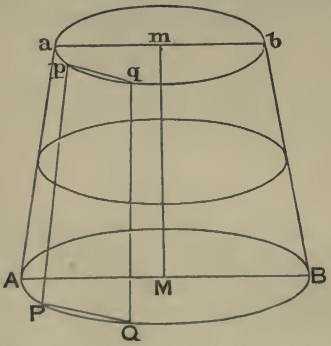
and vol′ of cone $= \frac{1}{3}\pi\rho^2\alpha$.

Cor′ (1)—Total surf′ of cone $= \pi\rho\sigma + \pi\rho^2 = \pi\rho\,(\rho + \sigma)$.

Cor′ (2)—$\sigma^2 = \rho^2 + \alpha^2$.

Def′—The part of a cone intercepted between two planes, parallel to its base, is called a **frustrum** of the cone, or a **truncated cone**. It may be considered as generated by the revolution of a symmetrical trapezium about the perpendicular to its parallel sides at their mid points.

THEOREM (3)—*The area of the curvilinear surface of a frustrum of a cone* (**s** *being one of the non-parallel sides of its generating trapezium, and* **r** *the radius of the circle mid-way between the ends of the frustrum*) *is* $2\pi \mathbf{r}\mathbf{s}$.



Let **AB ba** be a trapez′; **M**, **m** the mid p′ts of **AB**, **ab**; so that the frustrum is generated by the revolution of **AB ba** about **Mm**.

Let **PQ** be a side of a reg′r pol′ inscribed in the ⊙ generated by **AB**; **pq** the corresponding side of a sim′r and sim′ly placed pol′ in ⊙ generated by **ab**; so that **PQ qp** is a symmetrical trapez′.

Then area **PQ qp** $= \frac{1}{2}(\mathsf{PQ} + \mathsf{pq})$. dist′ between **PQ**, **pq**.

And the curvilinear surf′ is the Limit of the sum of such areas,

$$\text{i.e.} = \frac{\mathsf{Aa}}{2}\, L^t\, \Sigma\,(\mathsf{PQ} + \mathsf{pq}).$$

∴, ultimately, area curved surf′ $= \frac{1}{2}\mathsf{Aa}\,(2\pi\,\mathsf{MA} + 2\pi\,\mathsf{ma})$

$$= \frac{1}{2}\mathbf{s} \,.\, 4\pi\mathbf{r},$$

$$= 2\pi\mathbf{r}\mathbf{s}.$$

THEOREM (4)—*If* h *is the altitude of a frustrum of a cone;* R, r *the radii of its ends; then its volume is* $\frac{\pi h}{3}(R^2 + r^2 + Rr)$.

For the frust′ of the cone is the Limit of the regular inscribed frust′ of a pyr′. But vol′ frust′ pyr′ = sum of vol′s of three pyr′s of alt′ h and whose bases are the ends, and a mean prop′l between the ends.

Now these bases, in the case of the cone, are respectively πR^2, πr^2, and πRr.

Whence the Theorem follows.

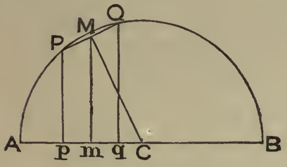
THE SPHERE.

Def′—If a semi-circle revolves about its diameter as a hinge, until it returns to its original position, the solid generated is called a **sphere**; and the centre and radius of the semi-circle are called the *centre* and *radius* of the sphere.

Note—By the nature of its definition every point on the surface of a sphere is equidistant from its centre.

Def′—A **frustrum of a sphere** is the part of a sphere which is intercepted between two parallel planes; and the circles of section of the planes are called the *bases* of the frustrum.

THEOREM (5)—*The surface of a sphere* (*radius* r) *is* $4\pi r^2$.



Let C be the centre, ACB the diam′ of the semi-⊙ which by turning round AB generates the sphere.

Let PQ be one of a number of consecutive equal ch′ds placed round the arc AB.

Draw CM ⊥ to, and ∴ bisecting PQ.

Let pmq be the projec′ of PMQ on ACB.

Then PQ, by its revolution round AB, generates a frustrum of a cone; and the curved surf′ of this frust′ is $2\pi Mm \cdot PQ$.

$$\text{But } PQ : pq = CM : Mm.$$

$$\therefore \quad Mm \cdot PQ = CM \cdot pq.$$

$$\therefore \quad \text{curved surf}' \text{ of frust}' = 2\pi CM \cdot pq.$$

Summing for all frustra, we have, since CM is same for all,

$$\Sigma\,(\text{curved surf}'\text{ of frust}') = 2\pi\,\mathrm{CM}\,.\,\Sigma\,(\mathrm{pq}) = 2\pi\,\mathrm{CM}\,.\,2\mathrm{r}.$$

Now the surf′ of the sphere is the Limit of all such surf′s, when the ch′ds are indefinitely increased in number.

But then CM becomes r.

$$\therefore\ \text{surf}'\text{ of sphere} = 4\pi\mathrm{r}^2.$$

Cor′ (1)—Since $4\pi\mathrm{r}^2 = \frac{2}{3}(2\pi\mathrm{r}^2 + 4\pi\mathrm{r}^2)$.

$\therefore$ surf′ of sphere $= \frac{2}{3}$ total surf′ of its circumscribing cyl′.

Cor′ (2)—The curved surf′ of a frustrum (called a *zone*) of a sphere = its thickness × circumf′ of generating ⊙ of sphere.

Note (1)—There is an essential difference between the curved surfaces of cylinders and cones, and those of spheres: the two former admit of *plane* flexible surfaces being rolled round them so as to coincide with them exactly, without creasing or folding; the latter do not. Otherwise: if the two former are imagined to be flexible, they can be flattened out into planes; but the latter cannot. Surfaces which may be conceived of as bent plates are called *developable surfaces*.

Note (2)—From the preceding characteristic property of developable surfaces the expressions [given in Theorems (1) and (2)] for the curved surfaces of cylinders and cones can be at once deduced: thus—

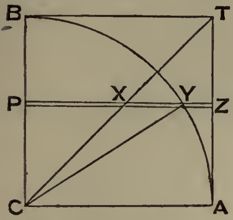
1°, let the curved surf′ of a cyl′ be cut along a position of its generating line; then, as in all positions the generating line remains || and equal to itself, if the cut surf′ is flattened out, it will become a rect′ (not crumpled)

whose breadth = length of generating line $= \alpha$,
and whose length = circumf′ of base $= 2\pi\rho$.
$\therefore$ area of curved surf′ of cyl′ $= 2\pi\rho\alpha$.

2°, let the curved surf′ of a cone be cut along a position of its generating line; then, as in all positions the generating line goes thro′ a fixed p′t and is of same length, if the cut surf′ is flattened out, it will become a sector of a ⊙ (not crumpled)

whose arc = circumf′ of base of cone $= 2\pi\rho$,
and whose rad′ = slant side of cone $= \sigma$.
$\therefore$ area of curved surf′ of cone $= \pi\rho\sigma$.

THEOREM (6)—*The volume of a sphere* (*radius* r) *is* $\frac{4}{3}\pi r^3$.



Let ABC be a quadrant of a ⊙, C being the centre of the complete ⊙.

Let the tang's at A, B meet in T, forming the sq′ CATB; and join CT.

Thro′ any p′t P in CB draw a ∥ to CA, cutting CT, arc AB, and AT, in X, Y, Z respectively; and suppose a ∥ drawn very near this last ∥.

Now if the whole fig′ is made to revolve about CB as a hinge, until it returns to its original position, AYB will generate a hemi-sphere, AT a cyl′, and CT a cone; and the ∥ˢ PXYZ will give a thin slice of the entire fig′. Join CY.

Then $PZ^2 = CA^2 = CY^2 = PY^2 + PC^2 = PY^2 + PX^2$.

$$\therefore \quad \pi . PZ^2 = \pi . PY^2 + \pi . PX^2.$$

$$\therefore \quad \Sigma(\pi . PZ^2) = \Sigma(\pi . PY^2) + \Sigma(\pi . PX^2).$$

Whence, diminishing the thickness, and increasing the number of the slices indefinitely, we get *in the Limit*,

$$\text{vol}' \text{ cyl}' = \text{vol}' \tfrac{1}{2} \text{ sphere} + \text{vol}' \text{ cone}.$$

$$\text{But vol}' \text{ cone} = \tfrac{1}{3} \text{ vol}' \text{ cyl}'.$$

$$\therefore \quad \text{vol}' \tfrac{1}{2} \text{ sphere} = \tfrac{2}{3} \text{ vol}' \text{ cyl}'.$$

$$\therefore \quad \text{vol}' \text{ sphere} = \tfrac{2}{3} \text{ vol}' \text{ of } \textit{its} \text{ circumscribing cyl}'$$

$$= \tfrac{2}{3} \times \pi r^2 \times 2r$$

$$= \tfrac{4}{3}\pi r^3.$$

Note—The volume of a sphere can be deduced from the area of its surface.

Imagine the surface of the sphere covered with a net-work of small areas: then the sphere may be considered as composed of the sum of all the pyramidal solids, whose common vertex is at the centre, and whose bases are these areas.

And each pyramidal solid tends to become a true pyramid as its base is taken smaller: i. e. tends to become one-third of its base into its altitude.

Now alt′ of each small solid = r,

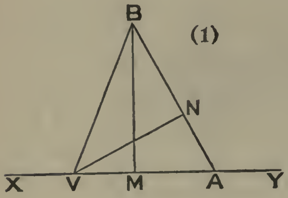
$$\therefore \quad \text{vol}' \text{ sphere} = \frac{r}{3} L^t \Sigma(\text{small areas})$$

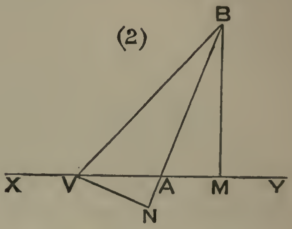
$$= \frac{r}{3} . \text{surf}' \text{ of sphere}$$

$$= \frac{4}{3}\pi r^3.$$

THEOREM (7)—*If in the plane of a triangle a line is drawn through one of its corners (considered as vertex) and if the plane revolves round this line, as a hinge, until it returns to its original position; then the volume generated by the triangle is equal to one-third of a cylinder whose base is the area generated by the base of the triangle, and whose altitude is the altitude of the trian*

Let VAB be the △, V being the vertex thro' which the axis of revolution XY passes; VN the alt'.





1°, let A lie in XY. Draw the ⊥ BM on VA: then BM may fall within the △, fig' (1), or without it, fig' (2).

Now vol' generated by △ VAB

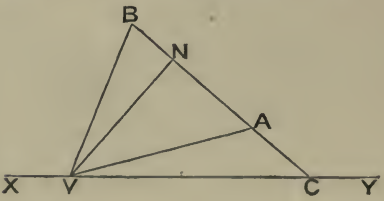
$= $ sum, or diff', of vol's due to △s VMB, AMB,

$$= \frac{\pi}{3} BM^2 (VM \pm AM)$$

$$= \frac{\pi}{3} BM^2 . VA$$

$= \frac{1}{3} VN . \pi BM . AB$, since $BM . VA = VN . AB$

$= \frac{1}{3} VN$. lat' surf' of cone generated by AB.



2°, let BA meet XY in C.

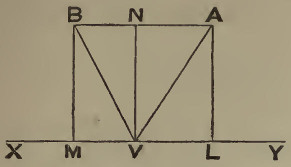
Then vol' generated by △ VAB

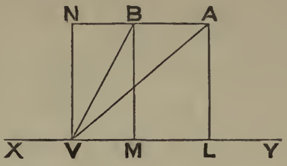
$= $ vol' due to △ VCB − vol' due to △ VCA,

$= \frac{1}{3} VN$ (area due to BC − area due to AC)

$= \frac{1}{3} VN$. area due to AB,

$= \frac{1}{3} VN$. lat' surf' of frust' of cone generated by AB.





3°, let AB be ∥ to XY.

Draw AL, BM ⊥s to XY.

Then, when the fig′ turns round XY,

vol′ cone due to △BMV = $\frac{1}{3}$ vol′ cyl′ due to MN.

vol′ cone due to △ALV = $\frac{1}{3}$ vol′ cyl′ due to LN.

∴ vol′ due to △VAB = $\frac{2}{3}$ vol′ due to ABML.

$= \frac{2}{3}\pi \mathrm{VN}^2 . \mathrm{AB}$.

$= \frac{1}{3}\mathrm{VN} . 2\pi\mathrm{VN} . \mathrm{AB}$.

$= \frac{1}{3}$ VN . lat′ surf′ of cyl′ generated by △VAB

∴, in all three cases, vol′ generated by △VAB = $\frac{1}{3}$ cyl′ whose alt′ is VN and base the surf′ generated by AB.

THEOREM (8)—*If a sector of a circle turns round (as a hinge) a diameter, which does not cut it; then, when it has returned to its original position, the volume generated is equal to one-third of a cylinder whose base is the surface generated by the arc of the sector, and whose altitude is the radius of the circle.*

For if we suppose equal ch′ds placed round the arc of the sector, and the ends of these ch′ds joined to the centre, we have a number of equal △s whose vertices are at the centre.

Now vol′ generated by each △ = $\frac{1}{3}$ a cyl′ whose base is the surf′ generated by its ch′d, and alt′ is alt′ of △.

Whence, taking the sum of all these, and passing to the Limit when the number of △s is indefinitely increased, we get the Theorem.

THEOREM (9)—*If a segment of a circle turns round (as a hinge) a diameter, which does not cut it; then, when it has returned to its original position, the volume generated is equal to one-sixth of a cylinder whose radius is the chord of the segment, and altitude the projection of this chord on the axis of revolution.*

Taking the fig′ of Theorem (5) (p. 67) let PQ be the seg′t, AB the diam′ round which it turns.

Then (CP, CQ being joined) vol′ generated by seg′t PQ

= vol′ by sector CPQ − vol′ by △CPQ

$= \frac{1}{3}\,r\,.$ surf′ gen′d by arc $PQ - \frac{1}{3}\,CM\,.$ surf′ gen′d by ch′d PQ

$= \frac{1}{3}\,r\,.\,2\pi\,pq\,.\,r - \frac{1}{3}\,CM\,.\,2\pi\,pq\,.\,CM$

$= \frac{2}{3}\,\pi\,pq\,(r^2 - CM^2)$

$= \frac{2}{3}\,\pi\,pq\,.\,\frac{PQ^2}{4}$

$= \frac{1}{6}\,\pi\,PQ^2.\,pq$

= one-sixth cyl′ whose rad′ is PQ, and alt′ pq.

THEOREM (10)—*The volume of a frustrum of a sphere is equal to a cylinder, whose base is half the sum of the bases of the segment, and altitude the thickness of the segment, together with a sphere whose diameter is the thickness of the segment.*

For, taking the fig′ of Theorem (5) p. 67,

Vol′ of frustrum generated by PpqQ

= vol′ gen′d by seg′t PQ + vol′ gen′d by trapez′ PMQqp.

$= \frac{1}{6}\,\pi\,.\,PQ^2\,.\,pq + \frac{1}{3}\,\pi\,.\,pq\,(Pp^2 + Qq^2 + Pp\,.\,Qq)$

$= \frac{1}{6}\,\pi\,.\,pq\,(2\,Pp^2 + 2\,Qq^2 + 2\,Pp\,.\,Qq + PQ^2)$

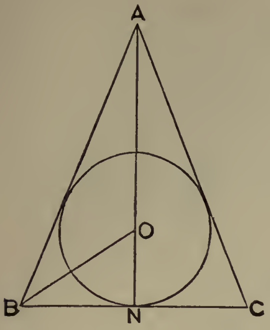
$= \frac{1}{6}\,\pi\,.\,pq\,\{3\,(Pp^2 + Qq^2) + pq^2\}$, since $PQ^2 = pq^2 + (Qq - Pp)^2$

$= \frac{1}{2}\,(\pi\,.\,Pp^2 + \pi\,.\,Qq^2)\,pq + \frac{1}{6}\,\pi\,.\,pq^3$

= a cyl′ whose base is ½ the sum of those gen′d by Pp and Qq, and whose alt′ is pq, + a sphere whose diam′ is pq.

THEOREM (11)—*If a sphere is inscribed in a cone; then,* V, S *being the volume and surface of the cone, and* V′, S′ *of the sphere*—

$$V : V' = S : S'.$$



Let AON be alt′ of cone, O being centre of sphere.

Take a section by a pl′ thro′ AN: this will give a △ ABC, and its in-⊙, centre O; and N will be p′t of cont′ of BC with in-⊙.

Let R be rad′ of base of cone, h its alt′, σ its slant side; and let r be rad′ of sphere.

Then, since BO bisects $\widehat{ABN}$,

$$\begin{aligned} BN : AB &= NO : OA. \\ \therefore\quad R : \sigma &= r : h - r, \\ \text{whence } Rh &= r(R + \sigma). \\ \text{Now } V : V' &= \tfrac{1}{3}\pi R^2 h : \tfrac{4}{3}\pi r^3; \\ \text{and } \therefore,\ \text{from above, } &= \pi R r (R + \sigma) : 4\pi r^3 \\ &= \pi R (R + \sigma) : 4\pi r^2 \\ &= S : S'. \end{aligned}$$

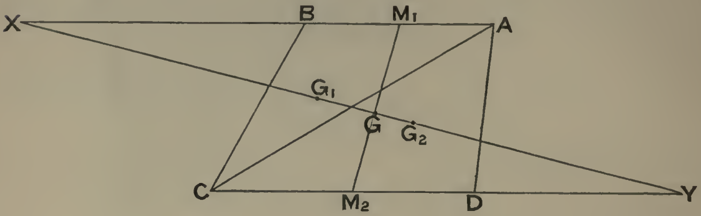
EXTENSION OF THE TERM MEAN CENTRE.

Since the mid point of a finite straight line is the *mean centre* (for equal multiples) of all pairs of points in the line, which are equidistant from that mid point, it is a natural extension of that term to define the mid point of a finite line as its **mean centre**. Again the *mean* centre of two finite lines is the mean centre of their mid points for multiples respectively proportional to their lengths. So also as the mean centres of all lines terminated by two sides of a triangle and parallel to a third lie in the median bisecting this third side; and similarly for lines parallel to another side; the **mean centre** of the area of a triangle is naturally defined to be its centroid.

Hence the mean centre of a polygon is the mean centre of the mean centres of the triangles into which it may be divided, for multiples respectively proportional to the areas of those triangles.

It follows from the preceding (by *E · R ·* pp. 112, 113) that the mean centre of the *area* a triangle is the same as the mean centre of its three corners, for equal multiples; and that the mean centre of the *perimeter* of a triangle is the in-centre of the triangle formed by joining the mid points of its sides.

As an example: let ABCD be a trapezium; G_1, G_2 the mean centres of the areas ABC, CDA; G the mean centre of the whole trapez′.



Then G_1 is mean centre of p′ts A, B, C for equal mult′s;
and G_2 ,, ,, ,, C, D, A ,, ,,
Take a as the mult′ for first set of p′ts.
b ,, ,, second ,,
where AB = a units of length, and CD = b units of length.
Let M_1, M_2 be the respective mid p′ts of AB, CD.

Then G is the mean centre of the four p′ts A, B, C, D for the respective mult′s $a + b, a, a + b, b$;

or of the two p′ts M_1, M_2 for mult′s $2a + b, a + 2b$.

$$\therefore \quad GM_1 : GM_2 = a + 2b : 2a + b,$$
$$= M_1X : M_2Y;$$

where X, in the production of AB, and Y in the production of CD, are such that

$$BX = CD, \text{ and } DY = AB.$$

$\therefore$ G is the intersection of the known lines M_1M_2 and XY.

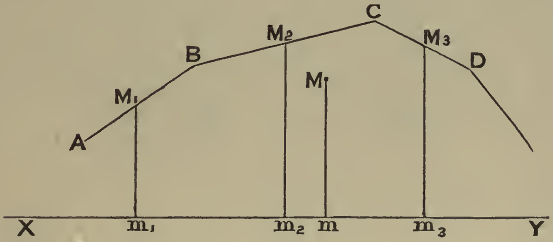
The foregoing process may be extended to solids, and will give us that the mean centre of a tetrahedron is the same as that of its four corners for equal multiples; and is geometrically constructed by joining two corners to the centroids of the opposite faces. The mean centre of a polyhedron is that of the mean centres of the tetrahedrons into which it can be divided, for multiples respectively proportional to the volumes of these tetrahedrons.

Finally, it will now be clear that the mean centres of a sphere, cylinder, and cone, are respectively the centre, mid point of axis, and point on axis three-fourths of its length from the vertex.

Note—In every case the geometrical mean centre has the same position as the centre of mass of the corresponding physical body. Now, as the positions of centres of mass is, in Dynamics, of the first importance, it follows that the geometrical theory of mean centres is also of great importance in relation to the geometry of Dynamics.

Def—A part of a polygon is called a **broken line.**

THEOREM (12)—(*Guldin's* or *Pappus'*)—*If a broken line and a fixed straight line, which it does not meet, are co-planar; and if their plane turns round the fixed line as a hinge, until it returns to its original position, the surface generated by the broken line is equal to the rectangle under two lines, one of which is the length of the broken line and the other the length of the path of its mean centre.*



Let ABCD &c. be a broken line, and XY a line co-planar with it, and round which it turns.

Let M_1, M_2, &c. be the mid p'ts of AB, BC, &c.; and m_1, m_2, &c. their respective projections on XY.

Then the area generated by a complete revolution of the broken line

$$= 2\pi\,(AB.\,M_1m_1 + BC.\,M_2m_2 + \&c.)$$

$$= 2\pi\,\Sigma\,(a.\,\alpha), \text{ where } AB = a,\ M_1m_1 = \alpha.$$

But, if M is the *mean centre* of the broken line,

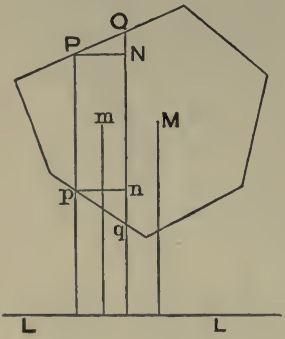
$$\text{Mm} = \frac{\Sigma(a.\alpha)}{\Sigma(a)} = \frac{\Sigma(a.\alpha)}{\text{ABCD \&c.}};$$

$$\text{and } 2\pi\,\text{Mm} = \text{path of } \textit{mean centre.}$$

∴ surf′ generated = ABCD &c. path of mean centre.

i.e. = rect′ under the line equal to ABCD &c. and the line equal to the path of the *mean centre.*

THEOREM (13)—(*Guldin's* or *Pappus'*)—*If the plane containing a polygon turns round any fixed line in itself* (*not cutting the polygon*) *as a hinge, until it returns to its original position, the volume of the solid ring generated by the polygon is equal to a cylinder whose base is the polygon and altitude the length of the path of its mean centre.*



All p′ts in the axis of revol′ are denoted by L.

Let a pair of contiguous ∥ pl′s PpL, QqL cut the pol′, intercepting the small slice PpqQ of the pol′.

Draw PN, pn ⊥ˢ to Qq.

Then, when pl′s are taken near enough together,

$$\text{vol}' \text{ generated by revol}' \text{ of PQqp} = \pi\text{PN}\,(\text{PL}^2 - \text{pL}^2)$$

$$= 2\pi\,\text{PN}\,.\,\text{Pp}\,.\,\frac{\text{PL} + \text{pL}}{2}$$

$$= 2\pi\,\text{mL}\,.\,\text{area slice.}$$

where m is the *mean centre* of Pn.

$$\text{And whole vol}' = 2\pi\,\Sigma\,(\text{mL}\,.\,\text{area slice}).$$

But, if M is *mean centre* of pol′, $\mathrm{ML} = \dfrac{\Sigma\,(\mathrm{mL}\,.\,\text{area slice})}{\text{area pol}'}$.

$\therefore$ vol′ generated by pol′ $= 2\,\pi\,\mathrm{ML}\,.$ area pol′

= vol′ cyl′ whose base is pol′ and alt′ is path of its *mean centre*.

Cor′—In the preceding Theorems suppose the broken line and pol′ each to be a reg′r pol′; and let the number of sides be increased indefinitely in each case: then these pol′s become ⊙s, and we get, by their revolution, what is called an *anchor ring*.

And 1°, surf′ of anchor ring $= 2\,\pi\mathrm{r}.\ 2\,\pi\mathrm{a} = 4\,\pi^2\,\mathrm{ra}$,

2°, vol′ „ „ $= \pi\mathrm{r}^2.\ 2\,\pi\mathrm{a} = 2\,\pi^2\mathrm{r}^2\mathrm{a}$,

where r = rad′ of ⊙, and a = dist′ of its centre from axis.

EXTENSION OF THE MODERN GEOMETRY OF LINES AND CIRCLES TO PLANES AND SPHERES.

Many of the Definitions and Theorems of Modern Plane Geometry will apply to Geometry in Space by making slight modifications of language. It will be sufficient to *state* the changes in many cases: the Student will have no difficulty in extending the proofs from the plane to the solid.

It may be taken as evident (or can be very easily proved) that—

1°, every section of a sphere by a plane is a circle;

2°, the line of section of two intersecting spheres is the circumference of a circle.

These being premised, we proceed to enunciate various definitions and theorems, analogous to those given in the *General Addenda* of *E*·*R*· In a few cases some steps of the reasoning will be indicated, but only where it seems to present any difficulty.

Cross-ratio.

THEOREM—*If four given planes have a line in common, and are cut by any line in points* A, B, C, D, *the cross-ratio of these four points is constant.*

For if any other line is drawn to cut the pl′s in A′, B′, C′, D′; and a pair of intersecting pl′s is drawn, so that AD is in one pl′, and A′ D′ in the other; then these pl′s will intersect in a line, and this line will cut the 4 given pl′s in 4 p′ts—say α, β, γ, δ.

Also $(ABCD) = (\alpha\beta\gamma\delta) = (A'B'C'D')$;
whence the theorem follows.

Def—The foregoing constant ratio is called the **cross-ratio** (or **anharmonic ratio**) of the four planes.

Cor′—The cross-ratio of 4 pl′s is the same as that of any 5th pl′ cutting them.

Centres and Axes of Similitude.

Def—If we take two circles with their pair of direct common tangents, and their pair of transverse common tangents, and let the whole figure revolve about the line of centres, as a hinge, until it returns to its original position, we shall have two spheres, enveloped by one cone, whose vertex is the external centre of similitude of the circles; and also enveloped by two cones, whose common vertex is the internal centre of similitude, and whose vertical angles are vertically opposite: the vertices of these cones are called respectively the **external** and **internal centres of similitude** of the spheres.

THEOREM (1)—*If a variable sphere touches two fixed spheres, the join of the points of contact goes through the external centre of similitude of the fixed spheres, when the contacts are of the same kind; and through the internal centre of similitude, when the contacts are of opposite kinds.*

THEOREM (2)—*The six centres of similitude of three spheres, taken two and two, are such that—*

1°, *the joins of the centre of each sphere with the internal centre of similitude of the other two, are concurrent;*

2°, *the external centre of similitude of any pair of spheres, and the two internal centres of similitude of the other two, are collinear:*

3°, *the three external centres of similitude are collinear.*

Def—Of the above lines the last is called the **external axis of similitude** of the three spheres; and the other three are called the **internal axes of similitude.**

Note—These four axes of similitude of three spheres are those of three circles, got by cutting the three spheres by a plane through their centres.

Def—If A, B, C, D, &c., are any points in space, and a, b, c, d, &c., points such that—

1°, Aa, Bb, Cc, Dd, &c., are concurrent in S; and,

2°, $SA : Sa = SB : Sb = SC : Sc =$ &c. $= K$, a constant;

then the systems A, B, C, D, &c., and a, b, c, d, &c., are said to be **similar** to each other, and **similarly situated** with respect to the point S; and S is called their *centre of similarity*.

It is evident that—

1°, any two parallel planes are similar, and similarly situated with respect to *any* point.

2°, any two spheres are similar, and similarly situated with respect to either of their centres of similitude.

Def—As a convenient abbreviation, we shall use the word **homothetic**—the English equivalent of the word *homothétique*, used by French writers—as an equivalent for the words—'similar and similarly situated': also—following the same writers—of two homothetic systems, each will be called the *homologue* of the other.

Note—A straight line, or a plane, is its own homologue with respect to any point in itself as a centre of similarity.

THEOREM (3)—*If each of two systems is homothetic to a third system the two systems are homothetic to each other; and the three centres of similarity are collinear.*

For (the fig′ can easily be supplied) let the system A, B, C, &c. be homothetic to each of the systems a, b, c, &c. and α, β, γ, &c.

Taking any two p′ts A, L of the 1st system, the corresponding p′ts of the other systems are a, l, and α, λ.

Then AL is ∥, and in a fixed ratio, to al;
and AL ,, ,, $\alpha\lambda$:
∴ al ,, ,, $\alpha\lambda$.

And sim′ly for all corresponding pairs of p′ts.

∴ system a, b, c, &c. is homothetic to system α, β, γ, &c.

Again let S_1, S_2, S_3 be the respective centres of similarity of the pairs of systems.

Then S_2S_3, considered as belonging to the 1st system, is its own homologue in the 2nd system, since it goes thro′ S_2, the centre of similarity of that system and the 1st.

So also it is its own homologue in the 3rd system, since it goes thro′ S_3, the centre of similarity of that system and the 1st.

∴ it must contain S_1, the centre of similarity of the 2nd and 3rd systems:

i.e. S_1, S_2, S_3 are collinear.

THEOREM (4)—*When four systems are homothetic, two and two, their six centres of similarity are co-planar.*

Call the systems (1), (2), (3), (4); and let S_1, S_2, S_3, be the respective centres of similarity of (1) and (2), (1) and (3), (1) and (4).

The pl′ $S_1 S_2 S_3$ is its own homologue with respect (1) and (2), since it contains their centre S_1.

It is also its own homologue in (1) and (3), since it contains their centre S_2.

∴ it is its own homologue in (2) and (3).

∴ it contains their centre of similarity:

i. e. the pl′ $S_1 S_2 S_3$ contains a 4th centre of similarity.

Sim′ly it can be shown to contain a 5th, and the 6th.

Note—These six co-planar centres of similarity are the corners of a complete quadrilateral, whose sides are the four axes of similarity of the four systems taken three and three.

Def′—Such a plane as the above is called a **plane of similitude** of the four systems.

THEOREM (5)—*Four spheres have eight planes of similitude.*

Four spheres have 12 centres of similitude, viz. 6 internal and 6 external.

Suppose a tetrahedron whose corners are the centres of the 4 sph′s.

On each face of this tet′ there are 6 cent′s of simil′, of which 3 are intern′ and 3 extern′.

Take S, one of these 6 cent′s of simil′ on one face.

S is on two axes of simil′ which are on one face of tet′, and also (being on an edge common to 2 faces) on two axes of simil′ which are on another face.

Taking these 4 axes two and two we get 4 pl′s, of which each, containing 5 cent′s of simil′, must also contain the 6th of the set.

Thus thro′ each of the 6 p′ts (like S) pass 4 pl′s of simil′.

But each of these pl′s contains an axis of similitude, and ∴ is common to 3 cent′s.

∴ there are 8 pl′s of simil′ in all.

Note—In the foregoing paragraphs, ‘*similar and similarly situated*’ is defined with respect to *one* centre of similarity; but the phrase is often explained, in more general terms, thus—

Def—Two curves are said to be **similar and similarly situated**, if any radius vector being drawn to one, from a fixed point S, and a *parallel* radius vector to the other, from another fixed point S′, these radii vectores are in a constant ratio: the points S, S′ are called *centres of similarity*.

Radical Planes.

Def'—That plane which is perpendicular to the line of centres of two spheres, and divides the distance between their centres (internally or externally) into segments, the difference of the squares on which is equal to the difference of the squares on their radii, is called the **radical plane** of the spheres.

THEOREM (1)—*The radical plane of two spheres is the Locus of points from which equal tangent lines can be drawn to the spheres.*

Cor'—When two spheres cut, their radical plane is the plane of their circle of section.

THEOREM (2)—*The radical plane of two spheres is the Locus of the centres of all spheres which cut the two spheres orthogonally.*

THEOREM (3)—*The radical planes of three spheres (whose centres are not collinear) taken two and two, have one line in common.*

Def'—This line is called the **radical axis** of the three spheres.

THEOREM (4)—*The six radical planes of four spheres (whose centres are not collinear) taken two and two, have one point in common.*

Def'—This point is called the **radical centre** of the four spheres.

Inversion.

Def'—If R is the radius of a sphere, C its centre, then every two points P, Q, collinear with C, and such that $CP.\ CQ = R^2$, are called **inverse points** with respect to that sphere.

THEOREM (1)—*The inverse of a plane is a sphere through the centre of inversion.*

THEOREM (2)—*The inverse of a sphere through the centre of inversion is a plane.*

THEOREM (3)—*The inverse of a sphere not through the centre of inversion is a sphere.*

THEOREM (4)—*Any radius vector, drawn from the centre of inversion as origin, to cut a sphere and its inverse, cuts them at supplementary angles.*

THEOREM (5)—*Two spheres cut at the same angle as their inverses.*

Cor' (1)—If two spheres (or a plane and a sphere) touch, their inverses touch.

Cor' (2)—If a sphere and its envelope are inverted, their inverses touch.

THEOREM (6)—*If* a, b, c, &c., *are the respective inverses of any number of points* A, B, C, &c., *in space*; *then* (O *being the centre of inversion*)—

1°, *for every two points* A, B,

$$ab : AB = R^2 : OA \,.\, OB;$$

2° *for every three points* A, B, C,

$$bc : ca = OA \,.\, BC : OB \,.\, CA;$$

3°, *for every four points* A, B, C, D,

$$bc \,.\, ad : ca \,.\, bd = BC \,.\, AD : CA \,.\, BD.$$

With similar changes, most of the Theorems on pp. 356, 357 of *Euclid Revised* will hold.

Poles and Polar Planes.

Def'—If X, Y are inverse points with respect to a sphere, on the same side of its centre, the plane through Y, perpendicular to XY, is called the **polar plane** of X with respect to the sphere, and X is called the **pole** of the plane through Y.

Note—When X is outside the sphere the polar plane cuts the sphere in a circle, whose centre is Y, and which is the curve of contact of the cone whose vertex is at X, and whose curved surface touches the sphere. When X, Y coalesce on the surface of the sphere, the polar plane is the tangent plane of the point of contact considered as pole.

THEOREM (1)—*When a plane goes through a fixed point, its pole lies on the polar plane of that point; and, conversely, when a point lies on a fixed plane, its polar plane goes through the pole of that plane.*

Cor'—If a variable cone circumscribes a sphere, so that 1°, the pl' of the curve of contact goes thro' a fixed p't, then its vertex lies on a fixed pl'; and, 2°, conversely, if its vertex lies on a fixed pl', then the pl' of its curve of contact goes thro' a fixed p't.

THEOREM (2)—*If a line cuts a sphere, and any point in the line is taken as a pole, then the point in which the polar plane cuts the line is the harmonic conjugate of the two points in which the line meets the surface of the sphere.*

Cor'—The pl' thro' any p't cuts the polar pl' of the p't in a line, which is the polar of the p't with respect to the circular section of the sphere made by the pl'.

THEOREM (3)—*If the Locus of a point is a fixed line, its polar-plane passes through another fixed line; and, conversely, if a plane passes through a fixed line, the Locus of its pole is a fixed line.*

Def'—Each of the two fixed lines is called the *polar line* of the other with respect to the sphere.

THEOREM (4)—*When four points are collinear, the cross-ratio of their four polar planes is the same as that of the points.*

SURFACE SPHERICS.

Def'—Every circle on a sphere, which is formed by a plane of section through the centre, is called a **great circle** of the sphere.

Def'—Every circle on a sphere, which is formed by a plane of section *not* through the centre, is called a **small circle** of the sphere.

Def'—The intersections of great circles form **spherical triangles**, **spherical quadrilaterals**, or (in general terms) **spherical polygons**—the intercepted arcs being called the *sides* of such polygons.

Note—Unless the contrary is expressly stated all spherical polygons are considered as having for their sides arcs of great circles.

Def'—The **join** (or the **spherical distance**) of two points on a sphere is the arc of the great circle drawn through them.

Def'—The **axis** of *any* circle on a sphere is the diameter of the sphere which is perpendicular to the plane of section through the circle; and the extremities of this axis are called the **poles** of the circle.

Note—It is easily seen that either intersection of two great circles is a pole of the join of their poles.

Def'—The **polar distances** of any circle are the joins of its poles to any point on the circumference of that circle; and the lesser of these, in the case of a small circle, is usually understood when *the* polar distance of a circle is spoken of: in the case of a great circle it is obvious that the polar distances are equal.

Def'—If ABC is a spherical triangle; and A′, B′, C′ the respective poles of BC, CA, AB; then *eight* such triangles as A′B′C′ exist; and of these there is *one* such that A′ and A, B′ and B, C′ and C, lie respectively towards the same parts of BC, CA, AB: this one is called **the polar triangle** of ABC.

Note—It can be easily shown that if A′B′C′ is the polar triangle of ABC, then, conversely, ABC is the polar triangle of A′B′C′.

Def'—The **angle between two spherical arcs** is considered to be represented by any one of the four following convertible measures—

1°, the dihedral angle between their planes:

2°, the plane angle between tangents to them at their point of intersection:

3°, the arc which (produced if necessary) they intercept on the great circle which the corner of the angle is pole:

4°, the spherical distance between their poles.

Def'—A portion of a surface of a sphere, enclosed by two semi-great-circles, is called a **lune**; and the angle between the circles is called the *angle of the lune.*

Note—Since area of lune : surface of sphere = angle of lune : 2π,

$$\therefore \text{ area of lune} = 2\,\mathsf{A}r^2;$$

where A = angle of lune, and r = radius of sphere.

The adequate treatment of surface spherics requires the aid of Spherical Trigonometry; but, starting from the definitions just given, a number of propositions can be proved, respecting spherical triangles, analogous to propositions proved for plane triangles. For example—

Euclid i. 4, 5, 6, 8, 13, 15, 18, 19, 20, 24, 25, 26, have their counterparts in the geometry of spherical triangles; so also many of the Problems of construction for plane triangles have their analogues in spherical geometry. However most of these are of little value; and what are useful will commonly be found in treatises on Spherical Trigonometry.

EXERCISES ON SOLIDS OF REVOLUTION.

THEOREMS.

1. Explain the following extract—

> 'That round Globe put into that round Drum, to touch it at the ends and all round, it is as precisely as if you clapt 2 into the inside of 3, not a jot more, not a jot less.'—CARLYLE'S *Dr. Francia.*

2. A system of spheres through the same circle are cut by a plane; show that the sections are a system of co-axal circles.

3. A system of spheres touch the same plane at the same point, and are cut by a plane; show that the sections are a system of co-axal circles.

4. Through four points one, and only one, sphere can be described.

5. If the diagonals of an octahedron are rectangularly concurrent, the projections of the point of concurrence on the faces are co-spheric.

6. If the diagonals of an octahedron are rectangularly concurrent, and its corners co-spheric, the sum of the squares on its edges, together with eight times the square on the distance of its circum-sphere from the point of concurrence of its diagonals, is equal to twenty-four times the square on the radius of its circum-sphere.

7. Deduce the second of the *Guldin* (or *Pappus*) theorems, in the particular case of a triangle, from Theorem (7).

8. Deduce the area of the surface of a sphere from the first of the *Guldin* (or *Pappus*) theorems.

9. A square and an equilateral triangle circumscribe a circle so that a side of each are in the same line, and the figure makes a complete revolution about that diameter of the circle which is perpendicular to this common line; show that the total area and volume generated by the square, are respectively mean proportionals between the total area and volumes generated by the triangle and circle.

10. Prove that the volume of a frustrum of a cone is equal to the sum of the volumes of a cylinder and of a cone, which have the same altitude as the frustrum, and the radii of whose bases are respectively half the sum and half the difference of the radii of the bases of the frustrum.

11. Prove that the volume of a shell contained between two spheres is equal to that of the frustrum of a cone the radii of whose bases are the diameters of the spheres, and whose altitude is the thickness of the shell.

12. Prove that when the side of a frustrum of a cone is equal to the sum of the radii of the bases, the height of the frustrum is twice the geometric mean of the radii; and the volume is equal to the total surface multiplied by the sixth of the height.

13. The total area (or volume) of the equilateral cylinder inscribed in, or circumscribed about, a sphere, is a mean proportional between the area (or volume) of that sphere, and the total area (or volume) of the equilateral cone respectively inscribed in, or circumscribed about, the sphere.

14. The distances of two points from the centre of a sphere, have the same ratio as their distances each from the polar plane of the other with respect to the sphere. (Extension of *Salmon's Theorem* in Plane Geometry: see *E. R.* p. 366)

15. Any three spheres can be inverted into three equal spheres. [*See E. R.* p. 357 (19)]

PROBLEMS.

1. Given the vertex of a cone and three of the points in which the hypotenuse of its generating triangle meets a given plane, construct for the point where its axis meets this plane.

2. If a triangle turns round one of its sides as a hinge, until it returns to its original position, find the ratio of the volumes generated by the two parts into which the triangle is divided by the join of the mid points of the other two sides.

3. A right-angled triangle is inscribed in a semi-circle, and the figure makes a complete revolution about the diameter: find the ratio of the volumes (or surfaces) generated by the triangle and semi-circle.

4. Through a given point draw a plane to touch two given spheres.

5. Draw a plane to touch three given spheres.

6. Circumscribe a sphere about a given tetrahedron.

NOTE—*Draw* $\perp^s$ *to two of the faces at their circum-centres: these will meet in the centre of the circum-sphere. Or bisect three of the edges by planes each* $\perp$ *to the edge it bisects: their common p't will be the centre of the circum-sphere.*

7. Inscribe a sphere in a given tetrahedron.

NOTE—*Bisect the dihed'* $\wedge^s$ *at the three edges of any one face: their common p't will be the centre of the in-sphere.*

8. Show how to find the volume of a frustrum of a cone by taking it as the difference between the complete cone and the smaller cone cut off by the smaller base of the frustrum.

9. Given a solid sphere, show how its radius may be found by the aid of a ruler and compasses only—the latter being used to transfer distances.

10. Construct a sphere *of given radius* so that it will—

1°, pass through three given points;

2°, pass through two given points and touch a given plane, or a given sphere;

3°, pass through a given point and touch two given planes, or two given spheres;

4°, touch three given planes, or three given spheres;

5°, pass through a given point and touch a given plane and a given sphere;

6°, touch two given planes and a given sphere;

7°, touch a given plane and two given spheres.

11. Construct a sphere so that it will—

1°, touch three given lines, one of them at a given point;

2°, pass through three given points and touch a given plane, or a given sphere;

3°, pass through two given points and touch two given planes, or two given spheres;

4°, pass through a given point and touch three given planes, or three given spheres;

5°, touch three given planes and a given sphere;

6°, touch a plane and three given spheres;

7°, touch two given planes and two given spheres.

12. Construct a sphere to touch four given spheres.

Note—The 16 *spheres of contact are given by either of the following constructions. See Gergonne's solution of the analogous Problem in Plane Geometry: E. R. p.* 350.

(1) *Draw a sph' cutting* 4 *sph's orthog'ly: let its pl's of intersec' with* 4 *sph's cut any one of the pl's, each of which contains* 6 *cent's of simil' of* 4 *sph's taken* 2 *and* 2 (*a pl' of* 3 *cent's being excluded*) *in* 4 *lines: thro' these lines draw tang' pl's to respective sph's; their p'ts of cont' will be p'ts of cont' of* 2 *sph's touching the* 4.

(2) *Take the poles with respect to the* 4 *sph's of any one of the pl's, each of which contains* 6 *of cent's of simil' of* 4 *sph's taken* 2 *and* 2 (*excluding a pl' of* 3 *cent's*) *join these poles to the radical cent' of the* 4 *sph's; joining lines will cut respective sph's in p'ts of cont' of* 2 *of sph's which touch the* 4.

G. F. WALKER: *Messenger of Mathematics*, vol' xi, p. 173. The second of these had been previously given, as the result of an analysis of the Problem, in the *Traité de Géométrie par Rouché et De Comberousse*, vol' ii, p. 266, 5th ed'.

EXERCISES ON SURFACE SPHERICS.

1. When is a spherical triangle self-polar?

2. Show that in a pair of polar triangles the sides of one are supplementary to the arcs which measure the respective opposite angles of the other.

3. Show that the three sides of a spherical triangle are together less than the circumference of a great circle.

4. By means of the last two Exercises prove that the three angles of a spherical triangle are together greater than two right angles, and less than six right angles.

NOTE—*The excess of the* 3 $\wedge^s$ *of a spherical* Δ *over* 2 rt. $\wedge^s$ *is termed 'the spherical excess' of the* Δ: *in a* Δ ABC *this is indicated by*

$$A + B + C - \pi = E.$$

5. Show that the area of a spherical triangle varies as its Spherical Excess.

NOTE—*Add the areas of the three lunes of which the* $\wedge^s$ *of the* Δ *are* $\wedge^s$.

6. If two spherical triangles have the three angles of the one respectively equal to the three angles of the other, prove that the three sides of the one are equal respectively to the three correspondingly situated sides of the other.

NOTE—*Take the polar* Δ^s, *and assume the analogue to Euclid* i. 8.

7. Prove that a cyclic spherical quadrilateral has the sum of one pair of its opposite angles equal to the sum of its other pair; and conversely.

8. Prove that the following triads of arcs, connected with a spherical triangle, are in each case concurrent.

1°, the three internal bisectors of the angles:

2°, two external bisectors of the angles, and the third internal bisector:

3°, the bisectors of the sides each of which is perpendicular to the side it bisects:

4°, the perpendiculars from the corners on the opposite sides.

9. If two diagonals of a spherical quadrilateral are quadrants, prove that the third diagonal is a quadrant.

10. If ABC is a spherical triangle; X, Y, Z the respective mid points of the sides opposite A, B, C; and if XY, produced both ways, meets AB produced in P, Q; prove that ZP, ZQ are each quadrants.

Hence solve this Problem—Given the base of a spherical triangle, and the length of the arc joining the mid points of the sides; find the Locus of its vertex.

11. Construct the in-circle, circum-circle, and ex-circles of a spherical triangle.

12. Given the base, and sum of base angles, of a spherical triangle, show that the internal bisector of the vertical angle goes through one fixed point, and the external bisector through another.

13. In a spherical triangle ABC, given AB, and A + B + C, show that the Locus of C is a small circle through A, B whose pole is known.

Hence solve this Problem—Given the base and area (cf. Exercise 5) of a spherical triangle, find the Locus of its vertex. (*Lexell's Locus*)

14. Prove that the arcs of great circles through the corners of a spherical triangle, bisecting its area, are concurrent. (*Steiner's Theorem*)

APPENDIX.

*THE GEOMETRICAL THEORY OF PERSPECTIVE IN SPACE.

Introduction—In what follows we shall use the term *cone* in its most general sense, viz. that in which the guiding curve of the generator may be any closed figure, whether curvilinear or rectilinear, or partly curvilinear and partly rectilinear. It is obvious that every tangent plane to a cone passes through the vertex.

Definition—The point of intersection of two straight lines will be termed the *cross* of the two straight lines.

We shall have occasion to make use of the following *Lemmas*.

Lemma i—If any number of points are collinear three by three, they are *all* collinear.

Lemma ii—If any number of straight lines, which are not all in the same plane, are concurrent two by two they are *all* concurrent.

Definition—Two plane figures in space are said to be *in perspective* when they are plane sections of the same cone.

There are two species of perspective—

1°, *direct*—when the sections are situated on the same side of the vertex.

2°, *inverse*—when the sections are situated on opposite sides of the vertex.

The vertex of the cone is called the *centre of perspective*, and the line of intersection of the planes of the sections is called the *axis of perspective*.

Note—When the perspective is direct it is clear that one figure exactly covers the other when viewed from the centre of perspective, hence the name.

Definition—When two plane figures in space are in perspective, we shall denote those points which are in perspective with each other by the term *corresponding points*; and also the line which joins two points in one figure and that which joins the corresponding points in the other, by the term *corresponding lines*.

---

* From a paper in *The Quarterly Journal of Mathematics*, No. 84, 1886; by Mr. ALEXANDER LARMOR, Fellow of Clare College, Cambridge.

Definition—When two figures are so related that to each line in one corresponds a definite line in the other, we shall denote the cross of two lines in the one and that of the corresponding lines in the other by the term *corresponding points.*

Note—When two plane figures *in space* are in perspective it follows that the cross of any two lines in one figure is in perspective with the cross of the corresponding lines in the other. For, draw the two planes passing through the centre of perspective and each of the two lines in the first figure: these planes contain, respectively, the corresponding lines in the second figure, and the two points in question therefore lie on their line of intersection. But when two rectilinear figures *in the same plane* have their angular points in perspective, it is clearly not necessary that the cross of any two chords of the perimeter of one should be in perspective with the cross of the corresponding chords of the perimeter of the other.

Definition—When this last condition *is* satisfied we shall speak of the two figures in the same plane as being in *complete* perspective *.

Definition—In the same manner two solids will be said to be in *complete* perspective when they may be regarded as made up of *corresponding* points (i. e. points in perspective) so related that the cross of any two lines in one solid is in perspective with the cross of the corresponding lines in the other.

THEOREM (1)—*If two triangles in space are so related that the joins of their corresponding vertices meet in a point, the crosses of their corresponding sides will lie on a straight line; and conversely.* (*Desargues*)

Let ABC, $\alpha\beta\gamma$ be the two triangles in different planes, not parallel.

The planes of ABC, $\alpha\beta\gamma$ intersect in a straight line (L, say).

But AB, $\alpha\beta$ being in one plane and not parallel must meet.

∴ their cross lies in L.

Similarly BC, $\beta\gamma$ meet in L; and so also CA, $\gamma\alpha$.

For the converse; we know by hypothesis that each pair of corresponding sides intersects.

∴ AB, $\alpha\beta$ intersect, and therefore lie in the same plane;

∴ $A\alpha$, $B\beta$ lie in the same plane and must intersect, if produced.

Similarly $B\beta$, $C\gamma$,, ,, ,,

,, $A\alpha$, $C\gamma$,, ,, ,,

∴ $A\alpha$, $B\beta$, $C\gamma$ meet in the same point. (*Lemma* ii.)

---

* By French writers they are called figures *in homology.*

THEOREM (2)—*If two triangles in the same plane are so related that the joins of their corresponding vertices meet in a point, the crosses of their corresponding sides will lie on a straight line; and conversely.* (*Desargues*)

Let ABC, abc be the two triangles, O their centre of perspective.

Let O′ be any point on the normal at O to the plane of the triangles, and let O′A, O′B, O′C meet the normals at a, b, c in a′, b′, c′ respectively.

Then O′ is the centre of perspective of the triangles ABC, a′b′c′.

Therefore, by (1), the crosses of their three pairs of corresponding sides lie on a straight line.

But ABC, abc are the orthogonal projections of these triangles on the original plane.

∴ the crosses of the three pairs of corresponding sides of these triangles lie on the same straight line.

For the converse; we know, by hypothesis, that the crosses of the three pairs of corresponding sides lie on a straight line (L, say).

Through this straight line draw a plane meeting the normals at a, b, c to the plane of the triangles, in a′, b′, c′ respectively.

Then ABC, a′b′c′ are two triangles in space, so related that the crosses of their three pairs of corresponding sides lie on the straight line L.

Therefore, by (1), they are in perspective.

Therefore ABC, abc, which are the orthogonal projections of these triangles on the original plane, are also in perspective. (Cf. *Euclid Revised*, p. 322)

THEOREM (3)—*If any two plane figures in space are in perspective, the crosses of all pairs of corresponding straight lines are collinear; and conversely, if two plane figures in space correspond, line for line, in such manner that the crosses of all pairs of corresponding straight lines are collinear, then the two figures are in perspective.*

For each pair of corresponding straight lines lies in a plane passing through the centre of perspective, and therefore will intersect if produced; their cross lies in the plane of each figure, and therefore on the line of intersection of their planes.

For the converse; take any three lines forming a triangle in one figure, and its corresponding triangle in the other; these triangles are so related that the crosses of their three pairs of corresponding sides are collinear.

Therefore, by (1), the joins of their corresponding vertices are concurrent.

Thus the joins of any three pairs of corresponding points are concurrent; whence the joins of all pairs of corresponding points are concurrent. (*Lemma* ii).

THEOREM (4)—*If any two figures in the same plane are in* **complete** *perspective, the crosses of all pairs of corresponding straight lines are collinear; and conversely, if any two figures in the same plane correspond, line for line, in such manner that the crosses of all pairs of corresponding straight lines are collinear, the two figures are in* **complete** *perspective.*

Take any three points in one figure and their corresponding points in the other. These are, respectively, the vertices of two triangles in perspective; therefore, by (2), the crosses of their three pairs of corresponding sides are collinear. Thus the crosses of any three pairs of corresponding straight lines in the two figures are collinear; whence (*Lemma* i) the crosses of all pairs of corresponding straight lines are collinear.

For the converse; take any three straight lines in one figure and their corresponding straight lines in the other: these form two triangles so related that the crosses of their three pairs of corresponding sides lie on a straight line.

Therefore, by (2), the joins of their three pairs of corresponding vertices are concurrent:

i. e. the joins of any three pairs of corresponding points in the two figures are concurrent.

Whence (*Lemma* ii) the joins of all pairs of corresponding points are concurrent.

THEOREM (5)—*If any two plane figures in space are in perspective, they will always remain so, however they may be turned about the line of intersection of their planes as axis.*

The criterion that they should always remain in perspective is, by (3), that the crosses of all pairs of corresponding straight lines should always be on the line of intersection of the planes of the figures; and this is clearly always satisfied, since it is satisfied in the initial positions of the figures.

THEOREM (6)—*If any two tetrahedrons are in perspective the crosses of corresponding edges lie in a plane.*

Let ABCD, abcd, be the two tetrahedrons, and let us denote the cross of *corresponding* edges AB, ab by AB . ab, &c.

Then, since ABC, abc are two triangles in perspective,

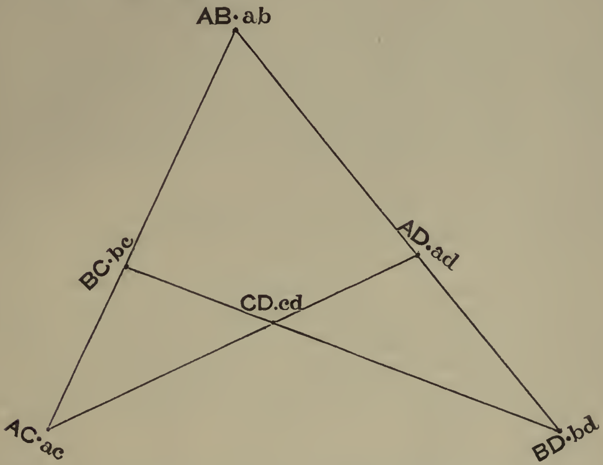
AB . ab, BC . bc, AC . ac are collinear.

Similarly, AB . ab, AD . ad, BD . bd ,, ,,

BC . bc, CD . cd, BD . bd ,, ,,

AC . ac, CD . cd, AD . ad ,, ,,

Thus the six crosses of corresponding sides lie three by three on four mutually intersecting straight lines, and must therefore also lie in the same plane, viz. the plane of the complete quadrilateral of which the six points are the vertices, as in the following figure.



THEOREM (7)—*If any two solids are in* complete *perspective, the crosses of corresponding straight lines are co-planar.*

For, any two corresponding straight lines lie in a plane containing the centre of perspective, and if any plane be drawn through the centre of perspective, it intersects the two solids in two figures that are in complete perspective, and therefore the locus of the crosses of corresponding lines in this plane is a straight line, by (4). Hence the locus of the crosses of *corresponding* lines in the two solids is a surface whose section by any plane through the centre of perspective is a straight line. This surface is clearly a plane; we shall call it *the plane of perspective* of the two solids.

THEOREM (8)—*If any three plane figures in space are in perspective, two and two, and if the perspectives be* (1°) *all direct, or* (2°) *one direct and two inverse, their three centres of perspective will be collinear, and their three axes of perspective will be concurrent.*

The axis of perspective of two plane figures (supposed ovals for clearness) is, by (3), the line of intersection of their planes, hence the three axes of perspective will meet in the point of intersection of the three planes of the figures, which proves the latter part of the proposition.

Again, by hypothesis, any two of the figures under consideration lie on a cone whose vertex is the corresponding centre of perspective, and every plane which touches both figures will be a tangent plane to this cone, and will therefore pass through the centre of perspective, provided, when the perspective is *direct* it has both figures on the same side, and when the perspective is *inverse* it has the figures on its opposite sides.

We can draw a pair of planes touching all the three given figures and having them all on one side of either plane, as is seen by making a plane roll round in contact with two of them until it touches the third; and we can in the same way draw three other pairs of planes touching all three figures in such a manner that each plane will have one figure on one side and the other two on its opposite side.

Now let us suppose that each pair of the given figures has two centres of perspective, one *direct* and one *inverse*, and there will be in all six centres of perspective, three *direct* and three *inverse*.

From the first paragraph it follows that the three *direct* centres of perspective lie on the line of intersection of the pair of planes touching all three figures, and having them all on one side of either plane; and again that each *direct* centre of perspective and the other two *inverse* centres lie on the line of intersection of the pair of planes touching all three figures, and having those two figures to which the *direct* centre belongs on one side of either plane of the pair, and the remaining figure on the other side.

Thus the six centres of perspective lie three by three on four straight lines and are therefore the vertices of a complete quadrilateral.

When only three centres of perspective exist the proposition enunciated still holds provided the perspectives are of the species therein specified.

THEOREM (9)—*If three solids are in* complete *perspective, two and two, and if the perspectives are* (1°) *all direct, or* (2°) *one direct and two inverse, their three centres of perspective will be collinear, and their three planes of perspective will pass through a straight line.*

For each pair of solids (considered ovals for clearness) has a common tangent cone whose vertex is the corresponding centre of perspective, and every plane which touches two of the solids will be a tangent plane to this cone and will therefore pass through the corresponding centre of perspective; provided, when the perspective is *direct*, it has both solids on the same side; and, when the perspective is *inverse*, it has the solids on its opposite sides.

As in (8) we see that we can draw eight planes to touch the three solids; and that when each pair of solids has two centres of perspective, one *direct* and one *inverse*, the three *direct* centres of perspective lie on a straight line and each direct with the other two inverse lies on another straight line; and when only three centres of perspective exist, the proposition enunciated still holds provided the perspectives are of the species therein specified.

Again, take any plane figure in one of the solids and the *corresponding* figure in each of the other solids. These figures are also plane, for a straight line corresponds to a straight line, and the surface corresponding to a plane may therefore be considered to be the Locus of a straight line which always intersects two given intersecting straight lines. The three axes of perspective of these three figures meet in a point, viz. the point common to their three planes. This point must obviously lie on each of the three planes of perspective. Thus the three planes of perspective have a series of points in common, which can only occur when they pass through the same straight line.

THEOREM (10)—*If three figures in the same plane are in* **complete** *perspective, two and two, and if the perspectives be* (1°) *all direct, or* (2°) *one direct and two inverse, their three centres of perspective will be collinear, and their three axes of perspective will be concurrent**.

This follows from (9) since we may consider the three figures to be sections, by a plane containing the three centres of perspective, of three solids that are in *complete* perspective two and two. A familiar example of this proposition is furnished by the case of three circles in the same plane. (Cf. *Euclid Revised*, p. 334)

---

* Two Theorems, due to *Chasles*—particular cases of this general Theorem—are enunciated in *Euclid Revised*: p. 329, Exercise 4. To render this part of the subject more complete, *Chasles*' proofs of them are here subjoined. The Student can easily supply the needful diagrams.

THEOREM (1)—*When three triangles, in perspective two and two, have the same axis of perspective, their three centres of perspective are collinear.*

Let **abc**, **a′b′c′**, **a″b″c″**, be three Δ^s, whose corresponding sides are concurrent, three and three, in the three collinear p′ts α, β, γ.

The three sides **ab**, **a′b′**, **a″b″** give (by joining their corresponding extremities) two Δ^s **aa′a″**, **bb′b″**, whose corners are, two and two, on three lines concurrent in γ.

Hence the three sides **aa′**, **aa″**, **a′a″**, of the first, meet respectively the three sides, **bb′**, **bb″**, **b′b″** of the second, in three collinear p′ts.

But these three p′ts are the centres of perspective of the three original Δ^s, taken two and two.

Hence the Theorem.

Cor'—Two homothetic Δ^s are two Δ^s in perspective, whose axis of perspective is at an infinite distance.

Hence, when three Δ^s are homothetic, two and two, their centres of similarity (which are the same as their centres of perspective) are collinear.

THEOREM (2)—*When three triangles, in perspective two and two, have the same centre of perspective, their three axes of perspective are concurrent.*

Let **abc**, **a′b′c′**, **a″b″c″**, be three Δ^s, whose corners are, three and three, on three lines concurrent in **S**, the common centre of perspective of the three Δ^s.

The three sides, **ab**, **a′b′**, **a″b″** form a Δ; and the three sides **ac**, **a′c′**, **a″c″** form a second.

These two Δ^s are in perspective, since the three sides of the one meet respectively the three sides of the other in three collinear p′ts **a**, **a′**, **a″**.

Hence the corners of these two Δ^s are, two and two, on three concurrent lines.

But these lines are precisely the axes of perspective of the three original Δ^s.

Hence the Theorem. (*Chasles ; Géométrie Supérieure*, pp. 283, 284)

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